PROBLEMS SESSIONS

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1. Exercise 1

1.1. Starting from the Hermitean matrix model

$$M_{ji} = \bar{M}_{ij}$$

and with potential

$$Z = \int dM \; e^{-N \, \mathrm{Tr}(V(M))} \; , \quad V(M) = M^2/2 - g\, M^3/3$$

write the diagrams for the free energy (connected vacuum diagrams) at order g^2. Collect them by powers of N and draw them on a surface with the appropriate genus.

1.2. Same question for the quartic model with potential

$$V(M) = M^2/2 - g\, M^4/4$$

1.3. One considers the Gaussian symmetric real matrix model. One considers $N \times N$ real symmetric matrices

$$M_{ij} = M_{ji}$$

with measure

$$dM = \prod_i dM_{ii} \prod_{i<j} dM_{ij}$$

and potential

$$Z = \int dM \; e^{-N \, \mathrm{Tr}(V_0(M))} \; , \quad V_0(M) = M^2/2$$

compute the Gaussian propagator and show that

$$\langle M_{ij} M_{kl} \rangle_0 = 1/(2N)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

1.4. Now consider the cubic and quartic symmetric matrix models, with the same actions as above. Write the diagrams for the free energy at order g^2 (if you like, you can go to order g^4). Show that one generates now also unoriented diagrams (i.e. fat graphs that can be drawn only on unoriented surfaces like the Klein bottle of the Moebius strip). What are the N factors associated to unoriented diagrams?

2. Exercise 2

2.1. What is the number $\mathcal{N}$ of (real) "gauge fixing" conditions for an $N \times N$ Hermitean matrix to be diagonal?

$$M = \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$

2.2. Start from a diagonal matrix M_0 and perturb it by an infinitesimal unitary transformation

$$M_0 \to M_\epsilon = U M_0 U^{-1} \quad , \quad U = 1 + \epsilon \quad , \quad \epsilon = -\epsilon^\dagger$$

with ϵ null on the diagonal.
How many (real) independent components has the matrix ϵ ? Compare with $\mathcal{N}$. Write the big $\mathcal{N} \times \mathcal{N}$ Jacobian derivative matrix $\mathcal{J}$ whose matrix elements are

$$\mathcal{J}_{ab} = \frac{\partial \text{ gauge fixing condition "}a\text{" for } M_\epsilon}{\partial \text{ component "}b\text{" for } \epsilon}$$

Show that the determinant of $\mathcal{J}$ is the square of the Vandermonde determinant for the diagonal matrix $M_0 = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$.

$$\det(\mathcal{J}) = \Delta(\lambda_1, \cdots \lambda_N)^2$$

2.3. Repeat theses calculations for the Real Symmetric Matrix ensemble and show that

$$\det(\mathcal{J}) = |\Delta(\lambda_1, \cdots \lambda_N)|$$

2.4. Show that in the large N limit, the density of eigenvalues for the Gaussian Hermitean matrix model is given by the Wigner semicircle law.

2.5. Repeat the calculation for the quartic random matrix model.

2.6. Deduce from the previous question the number of planar rooted quadrangulations with n faces.
(start by computing the resolvent)

3. Exercise 3

3.1. preliminary question. Show that for a generic potential $V(x)$ associated with the orthogonal polynomials $p_k(x)$ of degree k, the following relation holds for all k:

$$x p_k(x) = Q_{k,k+1} p_{k+1}(x) + Q_{k,k} p_k(x) + Q_{k,k-1} p_{k-1}(x)$$

We now focus on the Gaussian potential $V(x) = \frac{x^2}{2}$ and we note $m_k(x)$ the orthogonal polynomial of degree k satisfying $m_k(x) = x^k + \sum_{l=0}^{k-1} c_k^l x^l$.

3.2. Compute $m_0(x)$, $m_1(x)$, $m_2(x)$ and $m_4(x)$.

3.3. Prove that for all n, $x m_n(x) = m_{n+1}(x) + \alpha_n m_{n-1}(x)$. Then compute α_n.

3.4. Check that Hermite polynomials $H_n(x) \stackrel{\text{def}}{=} (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$ satisfy the previous relations.

4. Exercise 4

4.1. Let $F_{planar}(g)$ be the generating functional for planar quadrangulations

$$F_{planar}(g) = \lim_{N\to\infty} N^{-2} \log \left[\int dM \, \exp(-N \, tr(M^2/2 - g/4M^4)) \right]$$

Write its derivative as a function of the resolvent. Deduce that

$$F'_{planar} = \frac{d}{dg} F_{planar}(g) = 1/4 \int dx \; x^4 \; \rho(x)$$

with $\rho(x)$ the density of eigenvalues.

4.2. The explicit form for ρ has been found to be

$$\rho(x) = \frac{1}{2\pi} \, g \, (b^2 - x^2) \sqrt{a^2 - x^2}$$

with

$$a^2 = \frac{2\left(1 - \sqrt{1-12g}\right)}{3g} \;\; , \quad b^2 = \frac{\sqrt{1-12g}+2}{3g}$$

Compute F'_{planar} and shows that it has a singularity at $g = g_c$ of the form

$$F'_{planar}(g) = \text{regular part} \; + \; cst \; (g_c - g)^{3/2}(1 + \mathcal{O}(g_c - g))$$

You may use a formal calculational software (only if you spent too much time with pen calculations).

4.3. Alternatively, show that the third derivative of F_{planar} diverges when $g \to g_c = 1/12$ as

$$F'''_{planar}(g) \propto (g_c - g)^{-1/2}(1 + \mathcal{O}(g_c - g))$$

by looking at the behaviour of the corresponding x integral near the endpoint $x = a$ of the e.v. distribution.

4.4. Deduce that $F_{planar}(g)$ has a singularity as

$$(g_c - g)^{5/2}$$

and that the number of (unmarked) quadrangulations with K squares, N_K, scales as

$$N_K \propto 12^K K^{-7/2}$$

Hint, use the fact that you can write

$$N_K = \frac{1}{2i\pi} \oint dg \; F_{planar}(g) \; g^{-K-1}$$

with a small c.c.wise contour around the origin, and deform the contour around the singularity at g_c.

5. Exercise 5

5.1. The hyperbolic metric in the Poincaré disc[1] is ($z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$)

$$ds^2 = \frac{4\,dz\,d\bar{z}}{(1 - z\bar{z})^2}$$

The scalar curvature R in a conformal metric $ds^2 = dzd\bar{z}\exp(\phi(z,\bar{z}))$ is known to be (Δ is the standard Laplacian operator)

$$R = -4\,\exp(-\phi)\,\Delta\phi$$

Compute the scalar curvature R of the Poincaré metric. Can you show that the angle between two intersection curves at a given points are the same in the metric $g_0 = dzd\bar{z}$ and in any conformally equivalent metric $g_\phi = ds^2 = dzd\bar{z}\exp(\phi(z,\bar{z}))$?

5.2. Find an analytic transformation $z \to w = f(z)$ which maps the Poincaré disk $\mathbb{D}$ onto the upper half plane $\mathbb{H} = \{w : \mathrm{Im}(w) > 0\}$, with for instance $f(-\mathrm{i}) = 0$, $f(1) = 1$, $f(\mathrm{i}) = \infty$. Deduce the metric in the Poincaré half plane. Can you show that geodesics are the (half) circles orthogonal to the real line $\mathbb{R}$ (the "infinity")? Deduce that circles orthogonal to the unit circle are geodesics in $\mathbb{D}$.

5.3. Which are the transformations $w \to w'$ from $\mathbb{H} \to \mathbb{H}$ which leave the metric invariant?

5.4. Consider a point $\mathbf{x}$ on the boundary of $\mathbb{H}$ (the "infinity") and the set of geodesics that starts from $\mathbf{x}$. A circle $\mathcal{C}_{\mathbf{x}}$ tangent to $\mathbb{R}$ at $\mathbf{x}$ is called an *horocycle*.

Show that any such horocycle $\mathcal{C}_{\mathbf{x}}$ is orthogonal to the geodesics starting from $\mathbf{x}$.

What are the geodesics starting from ∞ and the horocycles $\mathcal{C}_\infty$ tangent to ∞?

5.5. Consider the "slice" $\mathbf{S}$ of $\mathbb{H}$ given by

$$\mathbf{S} = \{w : \mathrm{Re}(w) \in [a,b]\} \qquad a < b \text{ real}$$

and glue the boundaries of this slice into an "hyperbolic cylinder" $\mathbf{C}$ by identifying the points $a + \mathrm{i}y$ and $b + \mathrm{i}y$ with same imaginary part.

Find an analytic map $w \to z$ from $\mathbf{D}$ onto the unit disc $|z| < 1$ that maps ∞ onto the origin $z = 0$ and the infinity circle of $\mathbf{D}$, i.e. the real segment $[a,b]$ onto the unit circle.

Show the the resulting metric in the unit disk minus the origin $\mathbb{D}\backslash\{0\}$ is of the form

$$ds^2 = \frac{dzd\bar{z}}{|z|^2\,\log(1/|z|)^2}$$

What is the curvature in this metric?

What became the geodesics starting from ∞ and the horocycles $\mathcal{C}_\infty$ tangent to ∞ in this new punctured disk $\mathbb{D}_\star = \mathbb{D}\backslash\{0\}$?

Can you compute the relation between the length of an horocycle and the area encloded inside it?

Which transformations leave invariant this metric?

[1] actually discovered by Riemann...