Exercises

Exercise 1: The algebra R_Λ

We consider the algebra R_Λ

$$R_\Lambda = \{a + \ell b : a, b \in \mathbb{R}\} \qquad (a+\ell b)(c+\ell d) = ab + \ell^2 cd + \ell(ad+bc) \quad \text{with } \ell^2 = -\Lambda \in \{0, \pm 1\}.$$

For $\ell^2 = -1$ we can choose $\ell = \mathrm{i}$ and obtain the complex numbers.

1. Show that for $\ell^2 = 1$ one has

   $$\tfrac{1}{2}(1 \pm \ell) \cdot \tfrac{1}{2}(1 \pm \ell) = \tfrac{1}{2}(1 \pm \ell) \qquad (1 \pm \ell) \cdot (1 \mp \ell) = 0$$

   and that every element $a + \ell b \in R_\Lambda$ with $a, b \in \mathbb{R}$ can be expressed uniquely as

   $$a + \ell b = \tfrac{1}{2}(1+\ell)c + \tfrac{1}{2}(1-\ell)d \quad \text{with} \quad c, d \in \mathbb{R}.$$

   Compute c, d as functions of a, b.

2. Compute $\exp(\ell t) = \sum_{k=0}^{\infty} \frac{\ell^k t^k}{k!}$ for $t \in \mathbb{R}$ and $\ell^2 = -1$ and $\ell = \mathrm{i}$. Draw the geometric figure in the plane $\mathbb{R}^2$ which contains all points $\exp(\ell t)$ for $t \in \mathbb{R}$.

3. Compute $\exp(\ell t) = \sum_{k=0}^{\infty} \frac{\ell^k t^k}{k!}$ for $t \in \mathbb{R}$ and $\ell^2 = 1$ using 1. Draw the geometric figure in the plane $\mathbb{R}^2$ which contains all points $\exp(\ell t)$ for $t \in \mathbb{R}$.

4. Compute $\exp(\ell t) = \sum_{k=0}^{\infty} \frac{\ell^k t^k}{k!}$ for $t \in \mathbb{R}$ and $\ell^2 = 0$. Draw the geometric figure in the plane $\mathbb{R}^2$ which contains all points $\exp(\ell t)$ for $t \in \mathbb{R}$

5. Show that for $\ell^2 \in \{0, \pm 1\}$ one has $\exp(\ell t) \cdot \exp(\ell s) = \exp(\ell(s+t))$ for all $t, s \in \mathbb{R}$.

Exercise 2: The hyperbolic plane and Möbius transformations

We consider the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ The hyperbolic distance of two points $z, w \in \mathbb{H}^2$ is defined as

$$\cosh d(z, w) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}$$

To a matrix $M \in \mathrm{SL}(2, \mathbb{R})$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{R}, ad - bc = 1$$

we associate the Möbius transformation

$$T_M : z \mapsto \frac{az+b}{cz+d}$$

1. Show that the Möbius transformation for

   $$M = \begin{pmatrix} e^{w/2} & 0 \\ 0 & e^{-w/2} \end{pmatrix} \qquad w \in \mathbb{R}$$

   maps the imaginary axis $\{\mathrm{i}y : y > 0\}$ in the upper half plane to itself and that the hyperbolic distance between a point $\mathrm{i}y$ with $y > 0$ and its image $T_M(\mathrm{i}y)$ is $d(\mathrm{i}y, T_M(\mathrm{i}y)) = w$.

2. We consider the matrix

$$M = \begin{pmatrix} \cosh(\frac{w}{2}) & \sinh(\frac{w}{2}) \\ \sinh(\frac{w}{2}) & \cosh(\frac{w}{2}) \end{pmatrix} \quad \text{with} \quad w \in \mathbb{R}.$$

Determine all fixpoints of T_M. Show that T_M maps the half-circle $\{z \in \mathbb{C} : |z| = 1, \mathrm{Im}(z) > 0\}$ to itself and that the geodesic distance between a point p of the half circle and its image is $d(p, T_M(p)) = w$.

Exercise 3: Model spacetimes and isometry groups

We consider for $\ell^2 \in \{0, \pm 1\}$ the 2×2-matrices with entries in R_Λ and the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\circ = \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix}$$

1. Show that every matrix M that satisfies $M^\circ = M$ can be parametrised uniquely as

$$M(x) = \begin{pmatrix} x_3 + \ell x_1 & \ell(x_2 + x_0) \\ \ell(x_2 - x_0) & x_3 - \ell x_1 \end{pmatrix} \qquad \text{with} \quad x_0, x_1, x_2, x_3 \in \mathbb{R}.$$

and determine the set of points $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ for which $\det(M(x)) = 1$. Compute

$$\langle M(x), M(x) \rangle := -\det(\mathrm{Im}_\ell M(x) + \ell\, \mathrm{Re}_\ell M(x))$$

and show that $X_\Lambda := \{M \in \mathrm{Mat}(2 \times 2, R_\Lambda) : \det(M) = 1, M^\circ = M\}$ with the metric $\langle\, ,\, \rangle$ coincides with 3-dimensional Minkowski, de Sitter and anti-de Sitter space for $\ell^2 = 0$, $\ell^2 = -1$ and $\ell^2 = 1$.

2. For $\ell^2 = 0$ we consider the map

$$H : \mathbb{R} \times \mathbb{H}^2 \to \mathrm{Mat}(2 \times 2, R_\Lambda), \qquad H(t, z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{t\ell}{\mathrm{Im}(z)} \begin{pmatrix} -\mathrm{Re}(z) & |z|^2 \\ -1 & \mathrm{Re}(z) \end{pmatrix}$$

Show that H takes values in $X_0 = \{M \in \mathrm{Mat}(2 \times 2, R_0) : \det(M) = 1, M = M^\circ\}$ and determine a vector $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with $H(t, z) = M((x_0, x_1, x_2, 1))$. Show that for each $t > 0$, the vectors x that are obtained in this way are on the hyperboloid

$$H_t = \{x \in \mathbb{R}^3 : x_0^2 - x_1^2 - x_2^2 = t^2\}.$$

Determine the image of the upper imaginary axis $g = \{\mathrm{i}y : \mathrm{Im}(y) > 0\}$ on this hyperboloid.

3. Show that for $\ell^2 = 1$, every matrix $P \in \mathrm{SL}(2, R_\Lambda) = \{M \in \mathrm{Mat}(2 \times 2, R_\Lambda) : \det(M) = 1\}$ can be expressed uniquely as $P = \frac{1}{2}(1 + \ell)A + \frac{1}{2}(1 - \ell)B$ with $A, B \in \mathrm{SL}(2, \mathbb{R})$. Compute the product of P and $N = \frac{1}{2}(1 + \ell)C + \frac{1}{2}(1 - \ell)D$ with $C, D \in \mathrm{SL}(2, \mathbb{R})$ and give a group isomorphism $\phi : \mathrm{SL}(2, R_\Lambda) \to \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$.

4. We consider the matrices

$$J_0 = \tfrac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad J_1 = \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Compute $A(t) := \exp(t\ell J_0) = \sum_{k=0}^\infty (\ell t J_0)^k / k!$ and $B(t) := \exp(t\ell J_1) = \sum_{k=0}^\infty (\ell t J_1)^k / k!$ for $\ell^2 \in \{0, 1, -1\}$ and $t \in \mathbb{R}$. Show that $A(t), B(t) \in X_\Lambda$ for all $t \in \mathbb{R}$. Determine vectors $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ and $y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ with

$$M(x) = \begin{pmatrix} x_3 + \ell x_1 & \ell(x_2 + x_0) \\ \ell(x_2 - x_0) & x_3 - \ell x_1 \end{pmatrix} = A(t) \qquad M(y) = \begin{pmatrix} y_3 + \ell y_1 & \ell(y_2 + y_0) \\ \ell(y_2 - y_0) & y_3 - \ell y_1 \end{pmatrix} = B(t).$$