Quantum Structure of Spacetime and Gravity
COST QSPACE Training School, Belgrade 2016
Geometry with Fluxes (Peter Schupp), Exercises 1

Action MP 1405
Quantum Structure of Spacetime

1. Courant algebroid axioms

A Courant Algebroid consists of a vector bundle $E \to M$ with a bracket $[.,.]$, a non-degenerate fiber-wise inner product $\langle .,. \rangle$ and a bundle map ("anchor") $\rho : E \to TM$ subject to the following three axioms

$$[e,[e',e'']] = [[e,e'],e''] + [e',[e,e'']] \ , \quad \rho(e)\langle e',e' \rangle = 2\langle [e,e'],e' \rangle \ , \quad \rho(e)\langle e',e' \rangle = 2\langle e,[e',e'] \rangle$$

for $e, e', e'' \in \Gamma E$.

(a) The last axiom can be rewritten as $[e,e] = \frac{1}{2}\mathcal{D}\langle e,e \rangle$, where $D = \iota \circ \rho^T \circ d$ and $\iota : E^* \to E$ is the isomorphism induced by $\langle .,. \rangle$ and $\rho^T : T^*M \to E^*$ is the transpose of the anchor map.

(b) Polarizing the last two axioms (i.e. substitute $e' = e_1 + e_2$) you can show $\rho(e)\langle e_1, e_2 \rangle = \langle [e,e_1], e_2 \rangle + \langle e_1, [e,e_2] \rangle = \langle e, [e_1,e_2] + [e_2,e_1] \rangle$.

(c) Derive two further important properties from the axioms:

$$[e, f \cdot e'] = \rho(e)(f) \cdot e' + f \cdot [e,e'] \ , \quad \rho([e,e']) = [\rho(e),\rho(e')]_{\text{Lie}}$$

(Note that $\langle f \cdot e', e'' \rangle = f \cdot \langle e', e'' \rangle$ *for* $f \in C^\infty(M)$.*)*

(d) The bracket is neither antisymmetric, nor does it satisfy the Leibniz rule in the first slot. Show that instead: $[f \cdot e', e] = f \cdot [e',e] - \rho(e)f \cdot e' + \mathcal{D}\langle f \cdot e', e \rangle$.

2. Metric-twisted Dorfman bracket

It is a standard result that the Dorfman bracket

$$[Y + \eta, Z + \zeta] = \mathcal{L}_Y(Z + \zeta) - i_Z d\eta \qquad \text{with } Y, Z \in \Gamma(TM) \text{ and } \eta, \zeta \in \Gamma(T^*M)$$

can be twisted by the B-field leading to a bracket with an extra term involving the 3-form flux H. (You are welcome to review this construction as an exercise.) Instead of using the antisymmetric B, the Dorfman bracket can also be twisted by the symmetric metric g (or even more generally by $\mathcal{G} = g + B$). In order to preserve the Courant algebroid axioms, the pairing must also be modified. The metric-twisted bracket and pairing are

$$[Y + \eta, Z + \zeta]_g = e^{-g}([e^g Y + \eta, e^g Z + \zeta]) \text{ and } \langle Y + \eta, Z + \zeta \rangle_g = i_Y \zeta + i_Z \eta + 2g(Y,Z) \ ,$$

where $e^g Y \equiv Y + g(Y,.)$ and the metric g maps the vector field Y to the 1-form $g(Y,.)$.

(a) Compute $2g(\nabla_X Y, Z) := \langle X, [Y,Z]_g \rangle_g - \langle X, [Y,Z]_{\text{Lie}} \rangle_g$ to obtain a Koszul formula for the connection. *(Unlike the original Koszul formula, your result will also hold for a non-symmetric metric* $\mathcal{G} = g + B$*. See arXiv:1512.00207 [hep-th] for the result.)*

This is in fact an exercise in standard differential geometry (no fancy math needed). To proceed, first derive (or convince yourself) that $\mathcal{L}_X g(Y,.) = X.g(Y,.) - g(Y,[X,.])$. Using the Cartan identity $\mathcal{L}_X = i_X d + d i_X$ this then also gives $i_X dg(Y,.) = X.g(Y,.) - g(Y,[X,.]) - dg(Y,X)$.

(b) Verify that metricity of the connection is ensured by the third Courant algebroid axiom: Show $X.g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ using the result of 1(b) (previous exercise).

(c) A more challenging exercise: Show that ∇_X defined in 2(a) is indeed an affine connection.
Hint: Use the result of part 1(d) to show $\nabla_X fY = X(f) \cdot Y + f \cdot \nabla_X Y$.
Think about possible generalizations of the construction for arbitrary Courant algebroids (using the anchor map).