Lecture III: Coherent states, loops and effective actions in NC field theory

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cost Action MP 1405
Quantum Structure of Spacetime

H.S., arXiv:1606.00646

The (maximally SUSY) IKKT matrix model

Ishibashi, Kawai, Kitazawa, Tsuchiya 1996

$$S[X,\Psi] = -Tr\left([X^a,X^b][X^{a'},X^{b'}]\eta_{aa'}\eta_{bb'} \;+\bar\Psi\gamma_a[X^a,\Psi]\right)$$

$$X^a = X^{a\dagger} \in Mat(N,\mathbb{C})\,, \qquad a = 0,...,9 \qquad (N\to\infty)$$

Ψ ... Majorana-Weyl fermions

1) nonpert. def. of IIB string theory (on $\mathbb{R}^{10}$) (*IKKT*)
2) $\mathcal{N}=4$ SUSY Yang-Mills gauge thy. on "*noncommutative*" $\mathbb{R}^4_\theta$

Symmetries:

- gauge symmetry $X^a \to UX^aU^{-1}, \quad U\in U(\mathcal{H})$
- $SO(10)$ rotations $X^a \to \Lambda^a_b X^a$, similarly spinor
  translations $X^a \to X^a + c^a\mathbb{1}$
- SUSY

- pre-geometric; geometry emerges on solutions
- solutions = (typically) fuzzy spaces (="branes")

  fluctuations around solutions = physical fields
- quantization well-behaved because of maximal SUSY

  leads to interaction between branes:

  (non-local UV/IR mixing) $\equiv$ IIB supergravity in $\mathbb{R}^{10}$

  (as predicted in string theory)
- conjecture: dynamics of brane geometry (for suitable branes):

  $\rightarrow$ ("emergent") 4D gravity

  fuzzy S^4_N seems to work ! H.S. arXiv:1606.00769

Quantization of matrix models:

$$Z = \int dX^a d\Psi \, e^{-S[X]-S[\Psi]}$$

similarly for correlation functions

...non-perturbative!

- includes integration over geometries !
- probably ill-defined for generic models

  (UV/IR mixing $\rightarrow$ strongly non-local physics)
- well-behaved (only?) for IKKT model, $D = 10$ due to max. SUSY
- non-trivial gauge theory can arise from fuzzy extra dims

background solutions: generic fuzzy space

$$X^a \sim x^a: \ \mathcal{M} \hookrightarrow \mathbb{R}^{10}$$

fluctuation around background $X^a \to X^a + \mathcal{A}^a(X^a)$

expand bare action to $\mathcal{O}(\mathcal{A}^2)$:

$$S[X+\mathcal{A}] = S[X] + \frac{2}{g^2} Tr\Big(2\mathcal{A}^a(\Box + \mu^2)X_a + \mathcal{A}_a\big((\Box + \mu^2)\delta^a_b + 2i[\Theta^{ab}, .] - [X^a,[X^b,.]]\big)\mathcal{A}_b\Big)$$

with $i\Theta^{ab} = [X^a, X^b]$

e.g. one-loop effective action = Gaussian approx. around background

$$Z[X] = \int\limits_{\text{Gauss}} d\mathcal{A} d\Psi e^{-S[X+\mathcal{A},\Psi]} = e^{-\Gamma_{\text{eff}}[X]}$$

Coherent states & applications

recall fuzzy S^2_N:

$$[X^a, X^b] = i\varepsilon^{abc} X^c, \qquad X^a X_a = \frac{1}{4}(N^2 - 1) \; = R_N^2 \; .$$

$X^a = \pi_N(J^a)$ ... irrep of $SU(2)$ on $\mathcal{H} = \mathbb{C}^N$

functions on S^2_N ... $\mathcal{A}_N = End(\mathcal{H}) = \bigoplus_{l=0}^{N-1} (2l+1)$

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S^2 as group orbit: let $p \in S^2$ ... north pole

$$\begin{array}{ccc} SU(2) & \to & S^2 \\ g & \mapsto & g \cdot p =: x \end{array}$$

stabilizer $\mathcal{K} \subset SU(2) \;\Rightarrow\; S^2 \cong SU(2)/U(1)$

coherent states on S^2_N:

$|p\rangle \in \mathcal{H}_N$ ... highest weight vector in $\mathcal{H}_N$

def.

$$\begin{aligned} |x\rangle &= g_x \cdot |p\rangle, \qquad g_x \in SU(2) \; \text{... coherent states} \\ x^a &= \langle x|X^a|x\rangle \equiv \langle X^a\rangle \in S^2, \qquad x^a x_a = \tfrac{1}{4}(N-1)^2 =: r_N^2 \end{aligned}$$

$|x\rangle$ ... one-to-one correspondence to points x on S^2 (up to phase) coherent states are optimally localized, minimize uncertainty

$$
\begin{aligned}
\delta^2 &:= \sum_a \langle (X^a - \langle X^a\rangle)^2\rangle \\
&= \sum_a \langle p|X^a X^a|p\rangle - \langle p|X^a|p\rangle\langle p|X^a|p\rangle \\
&= R_N^2 - r_N^2 = \frac{N-1}{2} \\
&=: L_{NC}^2 \approx \frac{2}{N} R_N^2
\end{aligned}
$$

Exercise 9 (*challenge*): show that the highest weight states minimize the uncertainty δ^2.
Hint: for given state $|\psi\rangle$ consider the vector $\vec{x}(\psi) := \langle\psi|X^a|\psi\rangle$ and rotate it such that $\vec{x}(\psi)$ points along $\vec{e}_z$

overlap of coherent states:

$$
\begin{aligned}
|\langle x|y\rangle|^2 = \left(\tfrac{1+x\cdot y}{2}\right)^{N-1} \quad &\approx \quad \exp(-\tfrac{1}{4}\phi^2(N-1)) \;=\; \tfrac{1}{c_N}\,\delta_N(x,y) \\
\phi \quad &= \quad \measuredangle(x,y)
\end{aligned}
$$

char. angle $\phi_N = \frac{\pi}{\sqrt{N}}$
char. angular momentum $l \sim \sqrt{N}$

overcompleteness:

$$\mathbb{1}_{\mathcal{H}} = c_N \int_{S^2} dx |x\rangle\langle x|, \qquad c_N = \frac{\dim \mathcal{H}}{\mathrm{VolS}^2}$$

(by $SU(2)$ invariance)

trace of any operator $\mathcal{O} \in End(\mathcal{H})$

$$\mathrm{tr}\mathcal{O} = \frac{\dim \mathcal{H}}{\mathrm{VolS}^2} \int_{S^2} dx \langle x|\mathcal{O}|x\rangle$$

generalizes to any quantized coadjoint orbit relation with $\mathbb{R}^2_\theta$, Q.M: focus at north pole $p \in S^2$

rescale $R \to \infty$ s.t. $\theta = \frac{2R^2}{N} = const$

$$[X^i, X^j] = i\theta\epsilon^{ij} \quad + \mathcal{O}(\frac{1}{N})$$

coherent state at "origin" = north pole:

$$|0\rangle \equiv |p\rangle, \qquad a|0\rangle \sim (X_1 + iX_2)|0\rangle = 0 \qquad \text{highest weight state}$$

shifted (rotated) coherent states:

$$|x\rangle = U_x|0\rangle \quad \text{where} \qquad U_x = \exp(i\phi_i J^i), \qquad x^i = R\,\epsilon^{ij}\phi_j$$

localization:

$$\langle x'|x\rangle = e^{-\frac{i}{2\theta}x^i\varepsilon_{ij}x'^j} e^{-\frac{|x-x'|^2}{4\theta}}$$

covers area $A_N = \theta$

operators and symbols

For an operator $\mathcal{O} \in End(\mathcal{H})$, define *symbol* of $\mathcal{O}$ as

$$\mathcal{O}(x) = \langle x|\mathcal{O}|x\rangle$$

... de-quantization of $\mathcal{O}$, "semi-classical limit"

conversely:

$$\mathcal{O} = c_N \int_{S^2} dx \tilde{O}(x)|x\rangle\langle x|$$

in particular

$$\hat{Y}^l_m = c_N \int_{S^2} dx Y^l_m(x)|x\rangle\langle x|$$

however very delicate for large momenta, misleading in UV

wavefunctions and UV / IR sectors

- IR ("semi-classical") sector:

  non-local matrix elements decay at distances $|x-y| \sim L_{NC}$

$$\langle x|\mathcal{O}|y\rangle \approx \langle x|\mathcal{O}|x\rangle\, \langle x|y\rangle \;\approx\; \frac{1}{c_N}\langle x|\mathcal{O}|x\rangle\, \tilde{\delta}_N(x,y)$$

  i.e. $\mathcal{O}(x)$ varies slowly on uncertainty scale $\delta = L_{NC}$

max. angular momentum $l \leq \sqrt{N}$, uncertainty $\Delta^2 = L_{NC}^2$

optimally localized semi-classical function

$$|p\rangle\langle p| =: \; \frac{1}{c_N}\, \delta_N(X;p)$$

- UV sector:

most $\mathcal{O} \in End(\mathcal{H})$ have $l > \sqrt{N}$, not in semi-classical sector.

best described by *non-local* string states

$$\psi_{x,y} := |x\rangle\langle y| \qquad \in End(\mathcal{H})$$

most extreme "function" on S^2_N:

$$Y^{N-1}_{N-1} = |p\rangle\langle -p|,$$

has $l_{UV} = N$, maximally de-localized

most NC "functions" are non-local
$\Rightarrow$ non-local contributions in loops!

Quasi-coherent states

analogous (quasi-) coherent states exist on generic fuzzy spaces $\mathcal{M}_N$

L. Schneiderbauer HS arXiv:1601.08007; cf. Ishiki arXiv:1503.01230

Add a "point probe" brane p_x at $x \in \mathbb{R}^D$,
matrix Laplacian for background $\mathcal{M}_N \cup p_x$ with point brane:

$$\mathfrak{X}^a = \begin{pmatrix} X^a & 0 \\ 0 & x^a \end{pmatrix} \qquad \in End(\mathcal{H} \oplus \mathbb{C}), \qquad a = 1, \ldots, D$$

string sector = off-diagonal "functions" $\Phi \in End(\mathcal{H} \oplus \mathbb{C})$ connecting $\mathcal{M}_N$ with p_x, of the form

$$\Phi = \begin{pmatrix} 0 & \cdots & 0 & \\ \vdots & \ddots & \vdots & |\phi\rangle \\ 0 & \cdots & 0 & \\ & \langle\phi| & & 0 \end{pmatrix} \quad \in \mathcal{H} \oplus \mathcal{H}^T$$

where $|\phi\rangle \in \mathcal{H}$.

Consider energy (=Laplacian) of these string states:

$$\begin{aligned}\Box_{\mathfrak{X}}\Phi &= \textstyle\sum_a[\mathfrak{X}^a,[\mathfrak{X}^a,\Phi]] = \sum_a(\mathfrak{X}^a\mathfrak{X}^a\Phi + \Phi\mathfrak{X}^a\mathfrak{X}^a - 2\mathfrak{X}^a\Phi\mathfrak{X}^a) \\ &= \begin{pmatrix} \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \sum\limits_a (X^a - x^a)^2|\phi\rangle \\ \langle\phi|\sum\limits_a (X^a - x^a)^2 & 0 \end{pmatrix}\end{aligned}$$

... shifted harmonic oscillator for $|\phi\rangle$ on $\mathcal{M}$
(recall: $\mathcal{M}_N$ has structure of phase space in QM,
cf. $P^2 + Q^2 \equiv \sum X_i^2$ ... harmonic oscillator at origin)

$$\mathfrak{X}^a = \begin{pmatrix} X^a & 0 \\ 0 & x^a \end{pmatrix}, \qquad a = 1, \ldots, d$$

Laplacian for string sector reduces to

$$\Box_x = \sum_{a=1}^{d} (X^a - x^a)^2$$

consider quadratic form

$$\begin{aligned} \tfrac{1}{2}\mathrm{tr}(\Phi^\dagger \Box_{\mathfrak{X}} \Phi) = \langle\phi|\Box_x|\phi\rangle \quad &= \textstyle\sum_a (\Delta_\phi X^a)^2 + \sum_a (\langle\phi|X^a|\phi\rangle - x^a)^2 \\ &= \delta^2(\phi) + |\vec{\mathbf{x}}(\phi) - \vec{x}|^2 \\ &=: E(\vec{x}) \end{aligned}$$

Exercise 10 : Check this ! Here

$$\vec{\mathbf{x}}(\phi) = \langle\phi|\vec{X}|\phi\rangle$$

and

$$\delta^2(\phi) = \sum_a \langle\phi|(X^a)^2|\phi\rangle - \langle\phi|X^a|\phi\rangle^2 = \sum_a \langle\phi|(X^a - \mathbf{x}^a(\phi))^2|\phi\rangle$$

is the dispersion of the state ϕ.

Quasi-coherent states

Let $\vec{x}$ be a point in target space $\mathbb{R}^D$. Then the *quasi-coherent state(s)* at $\vec{x}$ are defined to be the ground state(s) Ψ of $\Box_x$, and their eigenvalue

$$\Box_x \Psi = E(\vec{x})\Psi$$

is the *displacement energy*.

Quasi-coherent states minimize

$\delta^2(\phi) + |\vec{\mathbf{x}}(\phi) - \vec{x}|^2 = E(\vec{x})$ = Dispersion + displacement^2

Perelomov coherent states are quasi-coherent states.

denote set of quasi-coherent states by $\mathcal{S}_E$ (possibly with cutoff prescription, e.g. upper bound on E)

Then

$$\mathcal{M} := \vec{\mathbf{x}}(\mathcal{S}_E) \; := \{\langle\psi|X^a|\psi\rangle; \quad \psi \in \mathcal{S}_E\} \quad \subset \mathbb{R}^D$$

provides a semi-classical picture of the (generic) fuzzy space.

example: fuzzy shere S_N^2

can show:

The quasi-coherent state at $\vec{x} \in \mathbb{R}^3$ coincides with the Perelomov coherent state on S_N^2 which is closest to $\vec{x} \in \mathbb{R}^3$.

and

$$\mathcal{M} = \vec{\mathbf{x}}(\mathcal{S}_E) = S^2$$

Measuring finite Quantum Geometries via Quasi-Coherent States

Compute $E(x)$ for all $x \in \mathbb{R}^D$.

For $x \in \mathcal{M}$, expect Hessian $H_{\mu\nu} = \nabla_\mu \nabla_\nu E$ to have $2n = \dim \mathcal{M}$ small eigenvalues, clearly separated from the remaining higher EV's (which measure the transversal separation).
The corresponding $2n$ eigenvectors of $H_{\mu\nu}$ define the tanget space of $\mathcal{M}$ at x.
Can **measure** the location of the brane $\mathcal{M} \subset \mathbb{R}^D$ by scanning target space $\mathbb{R}^D$ and looking for ("quasi-") minima of $E(\vec{x})$, by following these tangential directions, possibly with cutoff

Mathematica package BProbe,
https://github.com/lschneiderbauer/BProbe/tree/v0.9.0

string states

Iso Kawai Kitazawa hep-th/0001027; H.S., arXiv:1606.00646

$$\left|{x \atop y}\right) := \psi_{x,y} := |x\rangle\langle y| \qquad \in End(\mathcal{H})$$
$$\left({x \atop y}\right| := \psi^\dagger_{x,y} := |y\rangle\langle x|$$

momentum operators

$$\mathcal{P}^a\,\mathcal{O} := [X^a, \mathcal{O}],$$
$$\Box\,\mathcal{O} := \mathcal{P}^a\mathcal{P}_a\mathcal{O}$$

expectation values

$$\left({x \atop y}\right| \mathcal{P}^a \left|{x \atop y}\right) = \mathrm{tr}\,\psi_{y,x}[X^a, \psi_{x,y}] = \vec{\mathbf{x}}(x) - \vec{\mathbf{x}}(y)$$
$$\left({x \atop y}\right| \mathcal{P}^a\mathcal{P}_a \left|{x \atop y}\right) = \mathrm{tr}\,\psi_{y,x}[X^a, [X_a, .]]\psi_{x,y} = E_{xy}$$

$$\boxed{E_{xy} = (\vec{\mathbf{x}}(x) - \vec{\mathbf{x}}(y))^2 + 2\Delta^2}$$

energy of string state = length $^2+$ zero point energy general matrix elements

$$\begin{aligned}
\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right| \mathcal{P}^a\mathcal{P}_a \left|\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) &= \langle x|X^aX^a|x'\rangle\langle y'|y\rangle + \langle x|x'\rangle\langle y'|X^aX^a|y\rangle - 2\langle x|X^a|x'\rangle\langle y'|X_a|y\rangle \\
&\approx (2\Delta^2 + \vec{x}^2 + \vec{y}^2 - 2\vec{x}\vec{y}))\,\langle x|x'\rangle\langle y'|y\rangle \\
&\approx E_{xy}\,\langle x|x'\rangle\langle y'|y\rangle
\end{aligned}$$

nearly diagonal

good localization properties in both position and momentum !!

propagator

claim:

$$(\bar{\Box} + \mu^2)^{-1} := c_N^2 \int\limits_{\mathcal{M}} dxdy \left|\begin{smallmatrix}x\\y\end{smallmatrix}\right) \frac{1}{E_{xy} + \mu^2} \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right| \approx (\Box + \mu^2)^{-1}$$

is excellent approximation to the propagator

because:

$$\begin{aligned}
(\tilde{\Box} + \mu^2)^{-1}(\Box + \mu^2)\left|\begin{smallmatrix}x\\y\end{smallmatrix}\right) &= c_N^2 \int dx'dy' \left|\begin{smallmatrix}x'\\y'\end{smallmatrix}\right) \tfrac{1}{E_{x'y'}+\mu^2} \left(\begin{smallmatrix}x'\\y'\end{smallmatrix}\right| (\Box + \mu^2)\left|\begin{smallmatrix}x\\y\end{smallmatrix}\right) \\
&\approx c_N^2 \int dx'dy' \left|\begin{smallmatrix}x'\\y'\end{smallmatrix}\right) \tfrac{1}{E_{x'y'}+\mu^2} (E_{xy} + \mu^2)\langle x'|x\rangle\langle y|y'\rangle \\
&\approx \left|\begin{smallmatrix}x\\y\end{smallmatrix}\right)
\end{aligned}$$

completely regular since $E_{xy} \geq \Delta^2$

trace formula for $End(\mathcal{H})$

$$\boxed{\mathrm{Tr}_{End(\mathcal{H})}\mathcal{O} = \frac{(\dim\mathcal{H})^2}{(\mathrm{Vol}\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \, \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right|\mathcal{O}\left|\begin{smallmatrix}x\\y\end{smallmatrix}\right)}$$

proof:

rhs = unique functional on $End(End(\mathcal{H}))$ invariant under $G_L \times G_R$

= Tr

example:

$$
\begin{aligned}
\mathrm{Tr}_{End(\mathcal{H})}[X^a,[X_a,.]] &= \frac{N^2}{(\mathrm{VolS}^2)^2}\int\limits_{S^2\times S^2} dxdx'\mathrm{tr}(|x\rangle\langle x'|)(|x'^a - x^a|^2 + 2\Delta^2)(|x'\rangle\langle x|) \\
&= \frac{N^2}{(\mathrm{VolS}^2)^2}\int\limits_{S^2\times S^2} dxdx'(|x'^a - x^a|^2 + 2\Delta^2) \\
&= \frac{N^2}{\mathrm{VolS}^2}\int\limits_{S^2} dx(|x^a(e) - x^a(x)|^2 + 2\Delta^2) \\
&\approx \frac{1}{4}\frac{N^2(N^2-1)}{4\pi^2}\int\limits_{S^2} dx(|e_3 - x|^2 + O(\tfrac{1}{N})) \\
&= \tfrac{1}{2}N^2(N-1)^2(1+O(\tfrac{1}{N}))
\end{aligned}
$$

using $r_N^2 = x^a x_a = \frac{1}{4}(N-1)^2$ and $\Delta^2 \approx \frac{N}{2}$

good agreement with exact result:

$$
\mathrm{Tr}_{End(\mathcal{H})}[X^a,[X_a,.]] = \sum_{j=0}^{N-1} j(j+1)(2j+1) = \frac{1}{2}N^2(N^2-1).
$$

more generally:

for any smooth function f

$$
\begin{aligned}
\mathrm{Tr}_{End(\mathcal{H})} f(\Box) &= \frac{N^2}{(\mathrm{Vol}S^2)^2} \int_{S^2} dx \int_{S^2} dy\, f(R_N^2|x-y|^2 + 2\Delta^2) \\
&= \frac{N^2}{\mathrm{Vol}S^2} \int_{S^2} dx f(r_N^2|e_3 - x|^2 + 2\Delta^2) \\
&= 2\pi \frac{N^2}{\mathrm{Vol}S^2} \int_0^\pi d\vartheta \sin\vartheta f(r_N^2(1-\cos\theta)^2 + \sin^2\theta) + 2\Delta^2) \\
&= \frac{N^2}{2} \int_{-1}^1 du f(2r_N^2(1-u) + 2\Delta^2) \\
&\approx \int_0^N dj\, 2j f(j^2 + 2\Delta^2) \approx \sum_{j=0}^{N-1} (2j+1) f\big(j(j+1) + 2\Delta^2\big) \\
&= \mathrm{Tr}_{j_{\max}} f(\Box_g + 2\Delta^2) \\
&= \mathrm{Tr}_{j_{\max}} f(\Box_g)
\end{aligned}
$$

shift by $2\Delta^2 = N - 1$ negligible for $N \gg 1$

UV dominates!

works because $\mathrm{spec}(\Box) = \mathrm{spec}(\Box_g)$ also in UV

one-loop propagator for ϕ^4 on S^2_N

$$S[\phi] = \frac{1}{N}\mathrm{tr}\Big(\frac{1}{2}\phi(\Box + \mu^2)\phi + \frac{g}{4!}\phi^4\Big) \; = S_0[\phi] + S_{\mathrm{int}}[\phi]\;.$$

1-loop effective action (= Gaussian approx.)

$$\begin{aligned}\Gamma_{\mathrm{eff}}[\phi] &= S[\phi] + \tfrac{1}{2}\mathrm{Tr}_{End(\mathcal{H})}\log\Big(S''[\phi]\Big)\\ (\psi, S''[\phi]\psi) &= \tfrac{1}{N}\mathrm{tr}\Big(\psi(\Box + \mu^2)\psi + \tfrac{g}{3}\phi^2\psi^2 + \tfrac{g}{6}\psi\phi\psi\phi\Big)\end{aligned}$$

expanded:

$$\begin{aligned}\Gamma_{1-loop}[\phi] &= \mathrm{Tr}\log(.(\Box + \mu^2). + \tfrac{g}{3}.\phi^2. + \tfrac{g}{6}.\phi.\phi)\\ &= \mathrm{Tr}\log(\Box + \mu^2) + \mathrm{Tr}\Big(.\tfrac{1}{\Box + \mu^2}(\tfrac{g}{3}\phi^2. + \tfrac{g}{6}\phi.\phi)\Big) + O(\phi^4)\end{aligned}$$

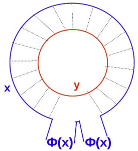
assume $\phi = \phi(X)$ slowly varying, IR regime

$\Rightarrow \phi\psi_{yx} \approx \phi(y)\psi_{yx}$ in string basis

$$\begin{aligned}
\mathrm{Tr}(.\phi^2.) &= \frac{N^2}{Vol(\mathcal{M})^2} \int_{\mathcal{M}\times\mathcal{M}} dxdy\, \mathrm{tr}(\psi_{y,x}\phi^2\psi_{x,y}) \\
&= \frac{N^2}{Vol(\mathcal{M})} \int_{\mathcal{M}} dx\, \langle x|\phi^2|x\rangle\ .
\end{aligned}$$

Similarly, "planar" contribution

$$\begin{aligned}
Tr(.\Box^{-1}\phi^2.) &= \frac{N^2}{Vol(\mathcal{M})^2} \int_{\mathcal{M}\times\mathcal{M}} dxdy\, \mathrm{tr}(\psi_{y,x}(\Box+\mu^2)^{-1}(\phi^2\psi_{x,y})) \\
&\approx \frac{N^2}{Vol(\mathcal{M})^2} \int_{\mathcal{M}\times\mathcal{M}} dxdy\, \frac{1}{r_N^2|x-y|^2+2\Delta^2+\mu^2}\mathrm{tr}(\psi_{y,x}\phi^2\psi_{x,y}) \\
&= \frac{N^2}{Vol(\mathcal{M}^2)} \int_{\mathcal{M}\times\mathcal{M}} dxdy\, \frac{1}{r_N^2|x-y|^2+\tilde{\mu}^2}\langle x|\phi^2|x\rangle \\
&= \frac{\mu_N^2}{Vol(\mathcal{M})} \int_{\mathcal{M}} dx\, \phi^2(x)
\end{aligned}$$



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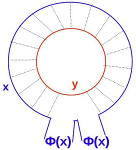
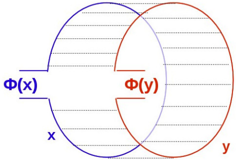


1-loop "planar" mass renormalization

$$
\begin{aligned}
\mu_N^2 &= \frac{N^2}{Vol(S^2)} \int_{S^2} dy \, \frac{1}{r_N^2|e-y|^2+\tilde{\mu}^2} \\
&= \frac{N^2}{2r_N^2} \int_0^\pi d\vartheta \sin\vartheta \frac{1}{(1-\cos\vartheta)^2+\sin\vartheta^2+\frac{\tilde{\mu}^2}{r_N^2}} \\
&= 2\int_{-1}^1 du \frac{1}{2-2u+\frac{\tilde{\mu}^2}{r_N^2}} \\
&\approx \sum_{j=0}^N \frac{2j+1}{j(j+1)+\mu^2}
\end{aligned}
$$

"nonplanar" contribution

$$\begin{aligned}
\mathrm{Tr}(.(\Box+\mu^2)^{-1}\phi.\phi) &= \frac{N^2}{Vol(\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\, \mathrm{tr}(\psi_{y,x}(\Box+\mu^2)^{-1}(\phi\psi_{x,y}\phi)) \\
&= \frac{N^2}{Vol(\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \langle x|(\Box+\mu^2)^{-1}\phi|x\rangle\langle y|\phi|y\rangle \\
&= \frac{N^2}{Vol(\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\, \frac{1}{r_N^2|x-y|^2+\tilde{\mu}^2}\phi(x)\phi(y)
\end{aligned}$$



one-loop quantum effective action:

$$S_{1-loop} \sim S_0 + \frac{g}{3}\frac{1}{\mathrm{Vol}(\mathcal{M})}\int\limits_{\mathcal{M}} dx \mu_N^2 \phi(x)^2 + \frac{g}{6}\frac{N^2}{\mathrm{Vol}(\mathcal{M})^2 \mathrm{r}_\mathrm{N}^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{\phi(x)\phi(y)}{|x-y|^2 + \frac{\tilde{\mu}^2}{r_N^2}} + O(\phi^4)$$

long-range non-locality from UV sector (UV/IR mixing)

applies to any compact fuzzy space

check for S^2_N: agrees with traditional mode expansion

$$S_{1-loop} = S_0 + \int \frac{1}{2}\Phi(\mu_N^2 - \frac{g}{12\pi}h(\tilde{\Box}))\Phi + o(1/N)$$

Chu Madore HS hep-th/0106205

where

$$h(L) = -\frac{1}{2}\int\limits_{-1}^{1} dt \frac{1}{1-t}(P_L(t) - 1) = \sum_{k=1}^{L}\frac{1}{k}$$

less transparent, requires asymptotics of 6J symbols etc.

Moyal-Weyl plane limit $\mathbb{R}_\theta^2$ $\qquad R^2 = r^2 R_N^2 = \frac{N\theta}{4}$

$$\begin{aligned}
\Gamma_{NP} &\approx \frac{gN^2}{6\mathrm{Vol}(\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{\phi(x)\phi(y)}{|x-y|^2+\mu^2} \\
&= \frac{g}{6\pi^2\theta^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{\phi(x)\phi(y)}{|x-y|^2+\mu^2}
\end{aligned}$$

where $\mathrm{Vol}\mathcal{M} = 4\pi R^2 = \pi N\theta$

plane wave basis $\phi(x) = \int \frac{d^2k}{2\pi} \phi_k(e^{ixk} + e^{-ikx})$.

$$\begin{aligned}
\Gamma_{NP} &\approx \frac{g}{6\pi^2\theta^2} \int d^2k \phi(k)^2 \int d^2z \frac{1}{|z|_g^2+\mu^2} e^{ik_i z^i} \\
&= \frac{g}{6\pi^2\theta^2} \int d^2k \phi(k)^2 \int d^2p \frac{1}{p_i p_j G^{ij}+\mu^2} e^{ik_i\theta^{ij}p_j}.
\end{aligned}$$

replacing $z^i = \theta^{ij}p_j$, and $G^{ij} = \theta^{ii'}\theta^{jj'}\delta_{i'j'}$

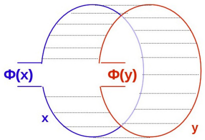
... familiar form in NCFT, IR divergence for $k \to 0$ from UV loop,

UV/IR mixing

significance of UV/IR mixing:

UV sector in loops = virtual long strings $|x\rangle\langle y|$

lead to long-range non-locality in $\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{\phi(x)\phi(y)}{|x-y|^2+\mu^2}$



interpret NCFT as (non-critical) string theory!
open strings beginning and ending on D-branes

universal, same on any fuzzy space $\mathcal{M}$, any dimension

accumulates at higher loops, inacceptable as fundamental theory

except in SUSY case: cancellations!

higher loops

t'Hooft double line formalism, ribbon graphs

lines labeled by positions x, preserved by propagators (!!)

much simpler than in ordinary QFT, directly in position space !

H.S., arXiv:1606.00646

(to be developed)

summary & outlook

- fuzzy spaces = noncommutative spaces embedded in $\mathbb{R}^D$
  realized by (finite-dim.) matrices $X^a, \; a = 1, ..., D$
- can realize generic geometries
- physical models naturally formulated as matrix models
- coherent states $|x\rangle$, "string states" $|x\rangle\langle y|$ useful
- UV/IR mixing understood due to long strings mediating interactions
- supersymm. IKKT model $\rightarrow$ mild non-locality (=IIB supergravity)
  4D gravity should (?!!) emerge on suitable branes (S^4_N)
  $\rightarrow$ candidate for theory of fundamental interactions including gravity