Fuzzy spaces and applications

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outline

1. Lecture I: basics
   • outline, motivation
   • Poisson structures, symplectic structures and quantization
   • basic examples of fuzzy spaces
     \((S^2_N, T^2_N, \mathbb{R}^4_\theta)\) etc.
   • quantized coadjoint orbits \((\mathbb{C}P^N_\mathbb{R})\)
   • generic fuzzy spaces; fuzzy \(S^4_N\), squashed \(\mathbb{C}P^2\) etc.
   • counterexample: Connes torus

2. Lecture II: developments
   • coherent states on fuzzy spaces (Perelomov)
   • symbols and operators, semi-class limit, visualization
   • uncertainty, UV/IR regimes on \(S^2_N\) etc.

3. Lecture III: applications
   • NCFT on fuzzy spaces: scalar fields & loops
   • NC gauge theory from matrix models
   • IKKT model
emergent gravity on $S^4_N$

**literature:**
These lectures will loosely follow the following:

- **introductory review:**
  H.S., “Noncommutative geometry and matrix models”. arXiv:1109.5521

Further related useful literature is e.g.

1 Lecture I: basics

Motivation, scope

Gravity ↔ quantum mechanics

General relativity (1915) established at low energies, long distances

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \]

Space-time: pseudo-Riemannian manifold \((M, g)\), dynamical metric \(g_{\mu\nu}\) describes gravity through the Einstein equations.

Is incomplete (singularities)

No natural quantization (non-renormalizable)

Q.M. & G.R. ⇒ break-down of classical space-time below \(L_{Pl} = \sqrt{\hbar G/c^3} = 10^{-33}\) cm

Classical concept of space-time as manifold physically not meaningful at scales \((\Delta x)^2 \leq L_{Pl}^2\)

⇒ expect quantum structure of space-time at Planck scale

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Standard argument: Consider an object of size \(\Delta x\).

Heisenberg’s uncertainty relation ⇒ momentum is uncertain by \(\Delta x \cdot \Delta p \geq \frac{\hbar}{2}\),

i.e. momentum takes values up to at least \(\Delta p = \frac{\hbar}{2\Delta x}\).

⇒ it has an energy or mass \(mc^2 = E \geq \Delta pc = \frac{\hbar c}{2\Delta x}\)

G.R. ⇒ \(\Delta x \geq R_{\text{Schwarzschild}} \sim 2G\frac{E}{c^2} \geq \frac{\hbar G}{c^3 \Delta x}\)

⇒ \((\Delta x)^2 \geq \hbar G/c^3 = L_{Pl}^2\)

More precise version:

(Doplicher Fredenhagen Roberts 1995 hep-th/0303037)

1.1 NC geometry

Replace commutative algebra of functions → NC algebra of “functions”

(Cf. Gelfand-Naimark theorem)

Inspired by quantum mechanics: quantized phase space

\([X^\mu, P_\nu] = i\hbar \delta^\mu_\nu\)
area quantization $\Delta X^\mu \Delta P^\mu = \hbar$ (Bohr-Sommerfeld quantization!)

NCG: not just NC algebra, but extra structure which defines the geometry

many possibilities

- Connes: (math) spectral triples
- here: alternative approach, motivated by physics, string theory, matrix models

1.2 Fuzzy spaces

**Definition 1.1.** Fuzzy space = noncommutative space $\mathcal{M}_N \hookrightarrow \mathbb{R}^D$

with intrinsic UV cutoff, finitely many d.o.f. per unit volume

similar mathematics & concepts as in Q.M., but applied to configuration space (space-time) instead of phase space

$$[X^\mu, X^\nu] = i \theta^{\mu\nu}$$

$\to$ typically quantized symplectic space

$\to$ area quantization $\Delta X^\mu \Delta X^\nu \geq \theta^{\mu\nu}/2$, finitely many d.o.f per unit volume

note:

- geometry from embedding in target space $\mathbb{R}^n$
  distinct from spectral triple approach (Connes)

- arises in string theory from D0 branes in background flux (“dielectric branes”)

- arises as nontrivial vacuum solutions in Yang-Mills gauge theory with large rank (“fuzzy extra dimensions”)

- condensed matter physics in strong magnetic fields (quantum Hall effect, monopoles (?) ...)

goal:

- formulate physical models (QFT) on fuzzy spaces
  study UV divergences in QFT (UV/IR mixing)

- find dynamical quantum theory of fuzzy spaces ($\rightarrow$ quantum gravity ?!)
1.3 Poisson / symplectic spaces & quantization

\[ \{.,.\} : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M}) \]  
... Poisson structure if

\[
\{f, g\} + \{g, f\} = 0, \quad \text{anti-symmetric}
\]

\[
\{f \cdot g, h\} = f \cdot \{g, h\} + \{f, h\} \cdot g \quad \text{Leibnitz rule / derivation,}
\]

\[
\{f, \{g, h\}\} = \text{cyclic} = 0 \quad \text{Jacobi identity}
\]

\[\leftrightarrow\] tensor field \(\theta^{\mu\nu}(x)\partial_\mu \wedge \partial_\nu\) with

\[
\theta^{\mu\nu} = -\theta^{\nu\mu}, \quad \theta^{\mu\nu} \partial_\mu \theta^{\rho\nu} + \text{cyclic} = 0
\]

assume \(\theta^{\mu\nu}\) non-degenerate

Then \(\text{exercise 1}\):

\[
\omega := \frac{1}{2} \theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu \quad \in \Omega^2 \mathcal{M} \quad \text{closed,}
\]

... symplectic form (=a closed non-degenerate 2-form)

\[
d\omega = 0
\]

examples:

- cotangent bundle: let \(\mathcal{M}\) ... manifold, local coords \(x^i\)
  
  \(T^* \mathcal{M}\) ... bundle of 1-forms \(p_i(x)dx^i\) over \(\mathcal{M}\)
  
  local coords on \(T^* \mathcal{M}\): \(x^i, p_j\)
  
  at point \((x^i, p_j)\) \(\in T^* \mathcal{M}\), choose the one-form \(\theta = p_i dx^i\). This defines a
  canonical (tautological) 1-form \(\theta\) on \(T^* \mathcal{M}\).
  
  The symplectic form is defined as \(\omega = d\theta = dp_i dx^i\)

- any orientable 2-dim. manifold
  
  \(\omega\) ... any 2-form, e.g. volume-form
  
  e.g. 2-sphere \(S^2\): let \(\omega = \text{unique } SO(3)\) -invariant 2- form

Darboux theorem:

 suppose that \(\omega\) is a symplectic 2-form on a \(2n\)- dimensional manifold \(\mathcal{M}\). for every \(p \in \mathcal{M}\) there is a local neighborhood with coordinates \(x^\mu, y^\mu, \mu = 1, \ldots, n\) such that

\[
\omega = dx^1 \wedge dy^1 + \ldots + dx^n \wedge dy^n = d\theta.
\]

so all symplectic manifolds with equal dimension are locally isomorphic
1.4 Quantized Poisson (symplectic) spaces

\((\mathcal{M}, \theta^{\mu\nu}(x))\) ... 2n-dimensional manifold with Poisson structure

Its quantization \(\mathcal{M}_\theta\) is given by a NC (operator) algebra \(\mathcal{A}\) and a (linear) quantization map \(\mathcal{Q}\)

\[ \mathcal{Q} : \mathcal{C}(\mathcal{M}) \rightarrow \mathcal{A} \subset \text{End}(\mathcal{H}) \]

\[ f(x) \mapsto \hat{f} \]

such that

\[ (\hat{f})^* = \hat{f}^* \]

\[ \hat{f} \hat{g} = \hat{f g} + o(\theta) \]

\[ [\hat{f}, \hat{g}] = i\{f, g\} + o(\theta^2) \]

or equivalently

\[ \frac{1}{\theta}([\hat{f}, \hat{g}] - i\{f, g\}) \rightarrow 0 \text{ as } \theta \rightarrow 0. \]

here \(\mathcal{H}\) ... separable Hilbert space

\(\mathcal{Q}\) should be an isomorphism of vector spaces (at least at low scales), such that

(“nice”) \(\Phi \in \text{End}(\mathcal{H}) \leftrightarrow \) quantized function on \(\mathcal{M}\)

cf. correspondence principle

we will assume that the Poisson structure is non-degenerate, corresponding to a symplectic structure \(\omega\).

Then the trace is related to the integral as follows:

\[ (2\pi)^n \text{Tr} \mathcal{Q}(\phi) \sim \int \omega^n \frac{\phi}{n!} = \int d^{2n}x \rho(x) \phi(x) \]

\[ \rho(x) = \text{Pfaff}(\theta^{-1}_{\mu\nu}) = \sqrt{\det \theta^{-1}_{\mu\nu}} \ldots \text{ symplectic volume} \]

(recall that \(\omega^n/n!\) is the Liouville volume form. This will be justified below)

Interpretation:

\[ \rho(y) = \sqrt{\det \theta^{-1}_{\mu\nu}} =: \Lambda_{\text{NC}}^{2n} \]

where \(\Lambda_{\text{NC}}\) can be interpreted as “local” scale of noncommutativity.

in particular: \(\dim(\mathcal{H}) \sim \text{Vol}(\mathcal{M}), \) (cf. Bohr-Sommerfeld)

examples & remarks:

- **Quantum Mechanics:**
  - phase space \(\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3 = T^*\mathbb{R}^3\), coords \((p_i, q^i)\),
  - Poisson bracket \(\{q^i, p_j\} = \delta^i_j\) replaced by canonical commutation relations
    \([Q^i, P_j] = i\hbar \delta^i_j\)
• reformulate same structure as $\mathbb{R}_h^2 = \text{Moyal-Weyl quantum plane}$

$$X^\mu = \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \text{Heisenberg C.R.}$$

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} \mathbf{1}, \quad \mu, \nu = 1, \ldots, 2, \quad \theta^{\mu\nu} = \hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathcal{A} \subset \text{End}(\mathcal{H}) \quad \ldots \text{functions on } \mathbb{R}_h^2$$

uncertainty relations $\Delta X^\mu \Delta X^\nu \geq \frac{1}{2} |\theta^{\mu\nu}|$

Weyl-quantization: Poisson structure $\{x^\mu, x^\nu\} = \theta^{\mu\nu}$

$$\mathcal{Q} : L^2(\mathbb{R}^2) \rightarrow \mathcal{A} \subset \mathcal{L}(\mathcal{H}), \quad (\text{Hilbert-Schmidt operators})$$

$$\phi(x) = \int d^2 k e^{ik\mu x^\mu} \hat{\phi}(k) \mapsto \int d^2 k e^{ik\mu X^\mu} \hat{\phi}(k) =: \Phi(X) \in \mathcal{A}$$

respects translation group.

interpretation:

$$X^\mu \in \mathcal{A} \cong \text{End}(\mathcal{H}) \quad \ldots \text{quantiz. coord. function on } \mathbb{R}_h^2$$

$$\Phi(X^\mu) \in \text{End}(\mathcal{H}) \quad \ldots \text{observables (functions) on } \mathbb{R}_h^2$$

• $\mathcal{Q}$ not unique, not Lie-algebra homomorphism

  (Groenewold-van Hove theorem)

• existence, precise def. of quantization non-trivial, $\exists$ various versions:

  – formal (as formal power series in $\theta$):
    always possible (Kontsevich 1997) but typically not convergent
  – strict (= as $C^*$ algebra resp. in terms of operators on $\mathcal{H}$),
  – etc.

need strict quantization (operators)

$\exists$ existence theorems for Kähler-manifolds (Schlichenmaier etal),
augmented-Kähler manifolds (= very general) (Uribe etal)

• semi-classical limit:

  work with commutative functions (de-quantization map).

  replace commutators by Poisson brackets
i.e. replace

\[ \hat{F} \rightarrow f = Q^{-1}(F) \]

\[ [\hat{F}, \hat{G}] \rightarrow i\{f, g\} \quad (+O(\theta^2), \text{ drop}) \]

i.e. keep only leading order in \( \theta \)

1.5 Embedded non-commutative (fuzzy) spaces

Consider a symplectic manifold embedded in target space,

\( x^a : M \hookrightarrow \mathbb{R}^D, \quad a = 1, \ldots, D \)

(not necessarily injective)

and some quantization \( Q \) as above. Then define

\[ X^a := Q(x^a) = X^a \dagger \in \text{End}(\mathcal{H}) \]

If \( M \) is compact, these will be finite-dimensional matrices, which describe quantized embedded symplectic space = fuzzy space.

**Definition 1.2.** A fuzzy space is defined in terms of a set of \( D \) hermitian matrices \( X^a \in \text{End}(\mathcal{H}), \ a = 1, \ldots, D, \) which admits an approximate "semiclassical" description as quantized embedded symplectic space with \( X^a \sim x^a : M \hookrightarrow \mathbb{R}^D. \)

aim: develop a systematic procedure to extract the effective geometry, formulate & study physical models on these.

1.6 The fuzzy sphere

1.6.1 classical \( S^2 \)

\( x^a : S^2 \hookrightarrow \mathbb{R}^3 \)

\[ x^a x^a = 1 \]

algebra \( \mathcal{A} = C^\infty(S^2) \) ... spanned by spherical harmonics \( Y^l_m = \text{polynomials of degree} \ l \) in \( x^a \)

choose \( SO(3) \)-invariant symplectic form \( \omega \), normalized as \( \int \omega = 2\pi N \)
1.6.2 fuzzy $S^2_N$

(Hoppe 1982, Madore 1992)

$S^2$ compact $\Rightarrow \mathcal{H} = \mathbb{C}^N, \mathcal{A}_N = \text{End}(\mathcal{H}) = \text{Mat}(N, \mathbb{C})$

would like to preserve rotational symmetry $SO(3)$

su(2) action on $\mathcal{A}_N$:

Let $J^a \ldots$ generators of $\text{su}(2)$,

$$[J^a, J^b] = i\varepsilon^{abc} J^c.$$ 

Let $\pi(N)(J^a) \ldots N-$ dim irrep of $\text{su}(2)$ on $\mathcal{H} = \mathbb{C}^N$ (spin $j = \frac{N-1}{2}$)

Define

$$\text{su}(2) \times \mathcal{A}_N \rightarrow \mathcal{A}_N$$

$$(J^a, \phi) \mapsto [\pi(N)(J^a), \phi]$$

decompose $\mathcal{A}_N$ into irreps of $SO(3)$:

$$\mathcal{A}_N = \text{Mat}(N, \mathbb{C}) \cong (N) \otimes (\bar{N}) = (1) \oplus (3) \oplus \ldots (2N-1) =: \{\hat{Y}^a_0\} \oplus \{\hat{Y}^1_1\} \oplus \ldots \oplus \{\hat{Y}^{N-1}\}.$$ 

... fuzzy spherical harmonics; UV cutoff in angular momentum!

Introduce Hilbert space structure on $\mathcal{A}_N = \text{Mat}(N, \mathbb{C})$ by

$$(F, G) := \frac{4\pi}{N} \text{Tr}(F^\dagger G)$$

corresponds to $L^2(S^2)$ with $(f, g) := \int_{S^2} f^* g$

-normalize the $\hat{Y}^l_m$ such that ONB,

$$(\hat{Y}^l_m, \hat{Y}^{l'}_{m'}) = 4\pi \delta^{l'l'} \delta_{mm'}$$

quantization map:

$$Q : \mathcal{C}(S^2) \rightarrow \mathcal{A}_N$$

$$(\hat{Y}^l_m \mapsto \begin{cases} \hat{Y}^l_m, & l < N \\ 0, & l \geq N \end{cases})$$

satisfies $Q(f^*) = Q(f)^\dagger$

embedding functions want $X^a \sim x^a$

note: $x^i : S^2 \hookrightarrow \mathbb{R}^3$ are spin 1 harmonics, $Y^1_{\pm 1} = x^1 \pm ix^2$ and $Y^1_0 = x^3$. Hence quantization given by $\hat{Y}^1_{\pm 1} = X^1 \pm iX^2$ and $\hat{Y}^1_0 = X^3$, i.e.

$$X^a := Q(x^a) = C_N \pi(N)(J^a)$$
for some constant $C_N$ (unique spin 1 irrep).

It follows

$$[X^a, X^b] = i C_N \varepsilon_{abc} X^c$$

fix radius to be 1,

$$\sum_{a=1}^{3} (X^a)^2 = C_N J^a J^a = C_N \frac{N^2 - 1}{4} 1,$$

cf. quadratic Casimir, implies

$$C_N = \frac{2}{\sqrt{N^2 - 1}} \approx \frac{2}{N}.$$  

correspondence principle $\rightarrow$ Poisson structure

$$\{x^a, x^b\} = C_N \varepsilon_{abc} x^c \approx \frac{2}{N} \varepsilon_{abc} x^c$$

which is of order $\theta \sim 2/N$, corresponds to $SU(2)$-invariant symplectic form

$$\omega = \frac{N}{4} \varepsilon_{abc} x^a dx^b dx^c =: N \omega_1$$

on $S^2$ with $\int \omega = 2\pi N$.

(unique closed and $SO(3)$ invariant volume form)

\[Exercise 2\]: check this by introducing local coordinates $x^1, x^2$ near north pole.

at north pole (NP): $\{x^1, x^2\} = \frac{2}{N}$

$\Rightarrow$ symplectic structure $\theta_{12} = \frac{1}{2}$ at NP

therefore:

[S^2_N is quantization of (S^2, N\omega_1)]

integral:

$$(2\pi) \text{Tr}(Q(f)) = \int_{S^2} \omega f$$

(only $\hat{Y}_0^0 \sim 1$ contributes).

$\exists$ inductive sequences of fuzzy spheres

$$\mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1} \hookrightarrow ... \hookrightarrow \mathcal{A} = C^\infty(S^2)$$
respecting norm and group structure (not algebra).

Realize $\hat{Y}_m^n = P_m^n(X)$ as totally symmetrized polynomials. Clearly the generators $X^a$ commute up to $1/N$ corrections, hence $Q(fg) \to Q(f)Q(g)$ for $N \to \infty$, for fixed quantum numbers. Thus

$$Q(fg) = Q(f)Q(g) + O(1/N),$$
$$Q(i\{f,g\}) = [Q(f), Q(g)] + O(1/N^2)$$

for fixed angular momenta $\ll N$.

For a fixed $S^2_N$, the relation with the classical case is only justified for low angular momenta, consistent with a Wilsonian point of view. (One should then only ask for estimates for the deviation from the classical case.)

**Example:** consider the coordinate "function"

$$X^3 = \frac{2}{\sqrt{N^2 - 1}} \text{diag}((N - 1)/2, (N - 1)/2 - 1, \ldots, -(N - 1)/2)$$

normalization such that the spectrum is essentially dense from $-1$ to $1$.

**Local Description:** near "north pole" $X^3 \approx 1$, $X^1 \approx X^1 \approx 0$

$$X^3 = \sqrt{1 - (X^1)^2 - (X^2)^2}$$
$$[X^1, X^2] = \frac{i}{\sqrt{C_N}} X^3 =: \theta^{12}(X) \approx \frac{2i}{N}$$

cf. Heisenberg algebra!

Quantum cell $\Delta A = \Delta X^1 \Delta X^2 \geq \frac{1}{N}$, total area $N\Delta A \sim 1$

$S^2_N$ consists of $N$ quantum cells

**Exercise 3:** Work out the “Jordan-Schwinger” (“2nd quantized”) realization for the fuzzy sphere, i.e. define

$$X^i := a^+_\alpha (\sigma^i)^{\beta}_\alpha a^\beta,$$
$$\alpha = 1, 2$$

for bosonic creation- and anihilation operators $[a^\alpha, a^+_\beta] = \delta^\alpha_\beta$ acting on the bosonic Fock space $F = \bigoplus_N F_N$, $F_N = a^+_{\alpha_1}, \ldots, a^+_{\alpha_N} |0\rangle$.

Show that the $X^i$ can be restricted to the $N$-particle sector $F_N$ specified by $X^i X_i \sim \hat{N} = a^+_\alpha a^\alpha = \text{const}$, and satisfy on $F_N$ the relations of a fuzzy sphere $S^2_N$. 

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1.7 Metric structure of the fuzzy sphere

*SO(3)* symmetry $\Rightarrow$ expect "round sphere"

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metric encoded in NC Laplace operator

\[ \Box : \mathcal{A} \to \mathcal{A}, \quad \Box \phi = [X^a, [X^b, \phi]] \delta_{ab} \]

*SO(3)* invariant: $\Box (g \triangleright \phi) = g \triangleright (\Box \phi) \Rightarrow \Box \hat{Y}^l_m = c_l \hat{Y}^l_m$

note: $\Box = C^2_N J^a J^a$ on $\mathcal{A} \cong (\mathcal{N}) \otimes (\bar{\mathcal{N}}) \cong (1) \oplus (3) \oplus \ldots \oplus (2N - 1)$

$\Rightarrow \Box \hat{Y}^l_m = C^2_N l(l + 1) \hat{Y}^l_m$

spectrum identical with classical case $\Delta_g \phi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu \nu} \partial_\nu \phi)$

up to cutoff

$\Rightarrow$ effective metric of $\Box =$ round metric on $S^2$

1.8 Fuzzy torus $T^2_N$

def. $U = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots \\ 0 & \ldots & 0 & 1 \\ 1 & \ldots & 0 \end{pmatrix}$, $V = \begin{pmatrix} 1 & e^{2\pi i \frac{1}{N}} & & & \\ & e^{2\pi i \frac{2}{N}} & & & \\ & & \ddots & & \\ & & & e^{2\pi i \frac{N-1}{N}} & \\ & & & & 1 \end{pmatrix}$ satisfy

$UV = qVU$, $U^N = V^N = 1$, $q = e^{2\pi i \frac{1}{N}}$

generate $\mathcal{A} = \text{Mat}(N, \mathbb{C})$ ... quantiz. algebra of functions on $T^2_N$

$\mathbb{Z}_N \times \mathbb{Z}_N$ action:

\[ \mathbb{Z}_N \times \mathcal{A} \to \mathcal{A}, \quad (\omega^k, \phi) \mapsto U^k \phi U^{-k} \]

$\mathcal{A} = \bigoplus_{n,m=0}^{N-1} U^n V^m$ ... harmonics

quantization map:

\[ Q : C(T^2) \to \mathcal{A} = \text{Mat}(N, \mathbb{C}) \]

\[ e^{in\varphi} e^{im\psi} \mapsto \begin{cases} q^{nm/2} U^n V^m, & |n|, |m| < N/2 \\ 0, & \text{otherwise} \end{cases} \]
satisfies
\[ Q(fg) = Q(f)Q(g) + O(\frac{1}{N}), \]
\[ Q(i\{f, g\}) = [Q(f), Q(g)] + O(\frac{1}{N^2}) \]

Poisson structure \( \{e^{i\varphi}, e^{i\psi}\} = \frac{2\pi}{N} e^{i\varphi}e^{i\psi} \) on \( T^2 \) \( \iff \{\varphi, \psi\} = -\frac{2\pi}{N} \)

integral:
\[ 2\pi Tr(Q(f)) = \int_{T^2} \omega_N f, \quad \omega_N = \frac{N}{2\pi} d\varphi d\psi = N\omega_1 \]

\( T_N^2 \) ... quantization of \((T^2, \omega_N)\)

metric on \( T_N^2 \) ? ... “obvious”, but need extra structure:

embedding \( T^2 \hookrightarrow \mathbb{R}^4 \) via \( x^1 + ix^2 = e^{i\varphi}, \quad x^3 + ix^4 = e^{i\psi} \)

quantization of embedding maps \( x^a \sim X^a : 4 \) hermitian matrices
\[ X^1 + iX^2 := U, \quad X^3 + iX^4 := V \]

satisfy
\[ [X^1, X^2] = 0 = [X^3, X^4] \]
\[ (X^1)^2 + (X^2)^2 = 1 = (X^3)^2 + (X^4)^2 \]
\[ [U, V] = (q - 1)UV \]

Exercise 4: derive this, and translate the last relation into commutation relations for \( X^a \)

Laplace operator:
\[ \Box \phi = [X^a, [X^b, \phi]]\delta_{ab} \]
\[ = [U, [U^\dagger, \phi]] + [V, [V^\dagger, \phi]] = 2\phi - U\phi U^\dagger - U^\dagger\phi U - (\%V) \]
\[ \Box(U^n V^m) = c(nq^2 + m\bar{q}^2) U^n V^m \sim c(n^2 + m^2) U^n V^m, \]
\[ c = -(q^{1/2} - q^{-1/2})^2 \sim \frac{1}{N^2} \]

Exercise 5: check this!
where
\[
[q]^n = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin(n\pi/N)}{\sin(\pi/N)} \sim n
\]
(“q-number”)

\[
\text{spec} \Box \approx \text{spec} \Delta_{T^2} \quad \text{below cutoff}
\]

therefore:

geometry of (embedded) fuzzy torus \( T^2_N \hookrightarrow \mathbb{R}^4 \) is \( \approx \) that of a classical flat torus

momentum space is compactified!

\([n]_q\)

compare: noncommutative torus \( T^2_\theta \)

Connes

Connes

\[
UV = qVU, \quad q = e^{2\pi i \theta}
\]

\[
U^* = U^{-1}, \quad V^* = V^{-1}
\]

generate \( \mathcal{A} \) ... algebra of functions on \( T^2_\theta \)

note: all \( U^n V^m \) independent, \( \mathcal{A} \) infinite-dimensional

in general non-integral (spectral) dimension, ...

for \( \theta = \frac{p}{q} \in \mathbb{Q} \): \( \infty \) -dim. center generated by \( U^n V^m q \)

fuzzy torus \( T^2_N \cong T^2_\theta / \mathcal{C}, \quad \theta = \frac{1}{N} \)

center \( \mathcal{C} \) ... infinite additional sector (meaning ??)

NC torus \( T^2_\theta \) very subtle, “wild”

fuzzy torus \( T^2_N \) “stable” under deformations

1.9 (Co)adjoint orbits

Let \( G \) ... compact Lie group with Lie algebra \( \mathfrak{g} = \text{Lie}(G) \cong \mathbb{R}^D \).

Then \( G \) has a natural adjoint action on \( \mathfrak{g} \) given by

\[
g \triangleright X = \text{Ad}_g(X) = g \cdot X \cdot g^{-1}
\]

for \( g \in G \) and \( X \in \mathfrak{g} \).
The (co-)adjoint orbit $\mathcal{O}[X]$ of $G$ through $X \in \mathfrak{g}$ is then defined as

$$\mathcal{O}[X] := \{ g \cdot X \cdot g^{-1} \mid g \in G \} \subset \mathfrak{g} \cong \mathbb{R}^D$$

$\mathcal{O}[X]$ is submanifold embedded in “target space” $\mathbb{R}^D$, invariant under the adjoint action.

Can assume that $X \in$ Cartan subalgebra, i.e. $X = H$ is diagonal.  
Is homogeneous space:

$$\mathcal{O}(H) \cong G/K_H$$

where $K_H = \{ g \in G : Ad_g(H) = 0 \}$ is the stabilizer of $H$.

Choose ONB $\{ \lambda_a, a = 1, \ldots, \dim \mathfrak{g} \}$ of $\mathfrak{g} \cong \mathbb{R}^D$,
structure constants 

$$[\lambda_a, \lambda_b] = i f_{ab}^c \lambda_c$$

→ Cartesian coordinate functions $x^a$ on $\mathbb{R}^D \ni X = x^a \lambda_a$,
defines function 

$$x^a : \mathcal{O}[X] \hookrightarrow \mathbb{R}^D$$

... characterize embedding of $\mathcal{O}[X]$ in $\mathbb{R}^D$, induce metric structure on $\mathcal{O}[X]$ 

### 1.9.1 Poisson structure on $\mathbb{R}^D$ and $\mathcal{O}[X]$: 

$$\{ x^a, x^b \} := f_{ab}^c x^c$$

(1)

extended to $C^\infty(\mathbb{R}^D)$ as derivation.

Jacobi identity is consequence of Jacobi identity for $\mathfrak{g}$

adjoint action of $\mathfrak{g}$ on itself ($=\mathbb{R}^D$) is realized through Hamiltonian vector fields

$$ad_{\lambda_a}[X] = [\lambda_a, X] = -i \{ x^a, X \}$$

Poisson structure is $G$- invariant

all Casimirs on $\mathfrak{g}$ are central, notably $C_2 \sim x_a x_b g^{ab}$

⇒ is not symplectic, but induces non-degenerate Poisson structure (symplectic structure) on $\mathcal{O}[X]$  

the $\mathcal{O}[X]$ are the symplectic leaves of $\mathbb{R}^D$. 

__________________________

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more abstract definition for symplectic structure:

$G$-invariant symplectic form on coadjoint orbit $O^*_\mu$ \hspace{1cm} ($\mu \in \mathfrak{g}$ ... weight)

\[ \omega_\mu(\hat{X}, \hat{Y}) := \mu([X,Y]) \]

where $\hat{X}$ ... vector field on $\mathfrak{g}^*$ given by action of $X \in \mathfrak{g}$ on $\mathfrak{g}^*$.

... an antisymmetric, non-degenerate and closed 2-form on $O^*_\mu$.

(Kirillov-Kostant-Souriau)

Example: sphere $S^2_N$

$G = SU(2)$, generators $\lambda_1, \lambda_2, \lambda_3 =$ Pauli matrices

coadjoint orbit through

\[ \lambda_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{su}(2) \]

stabilizer = $U(1)$

$S^2 = \mathcal{O}[\lambda_3] \cong SU(2)/U(1)$

Poisson bracket on $\mathbb{R}^3 = \mathfrak{su}(2)$

\[ \{x_a, x_b\} = \epsilon_{abc} x_c \]

respects $R^2 = x_a x^a$, symplectic leaves = $S^2$.

Example: complex projective space $\mathbb{C}P^2$

$G = SU(3)$, generators $\lambda_a =$ Gell-Mann matrices

coadjoint orbit through

\[ \lambda_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathfrak{su}(3) \]

stabilizer = $SU(2) \times U(1)$

$\mathbb{C}P^2 = \mathcal{O}[\lambda_8] \cong SU(3)/SU(2) \times U(1)$

Note:

$X := 2\sqrt{3} \lambda_8$ satisfies $(X + 1)(X - 2) = 0$

i.e. only two different eigenvalues
hence $X$ determines a rank 1 projector

$$P := \frac{1}{3}(X + 1) \in \text{Mat}(3, \mathbb{C})$$

satisfies

$$P^2 = P, \quad \text{Tr}(P) = 1$$

hence $P$ can be written as

$$P = |z^i \rangle \langle z^i|$$

where $(z^i) = (z^1, z^2, z^3) \in \mathbb{C}^3$, normalized as $\langle z^i | z^i \rangle = 1$.

Such projectors are equivalent to rays in $\mathbb{C}^3 \rightarrow$ conventional description of $\mathbb{C}P^2$ as $\mathbb{C}^3/\mathbb{C}^* \cong S^5/U(1)$.

Poisson bracket on $\mathbb{R}^8 = \mathfrak{su}(3)$

$$\{x_a, x_b\} = f_{abc} x_c$$

The embedding of $\mathbb{C}[X] \subset \mathbb{R}^8 \cong \mathfrak{su}(3)$ is described as follows: characteristic equation $X^2 - X - 2 = 0$ is equivalent to

$$\delta_{ab} x^a x^b = 3, \quad d_{abc} x^a x^b = x^c. \quad (2)$$

where $d_{abc}$ is the totally symmetric invariant tensor of $SU(3)$.

Exercise 6: derive the relations (2) using $\lambda_a \lambda_b = \frac{2}{3} \delta_{ab} + \frac{1}{2} (if_{abc} + d_{abc}) \lambda_c$

analogous construction for $\mathbb{C}P^n$:

$$\mathbb{C}P^n \cong O(\lambda) \cong SU(n + 1)/(SU(n) \times U(1))$$

is adjoint orbit of $SU(n + 1)$ through maximally degenerate generator

$$\lambda \sim \text{diag}(-1, -1, ..., -1, n)$$

up to normalization.
1.9.2 Functions on $O(\Lambda)$ & decomposition into harmonics:

$G$ acts on $O(\Lambda)$
$
\rightarrow$ decompose classical algebra of polynomial functions on $O(\Lambda)$:

$$Pol(O(\Lambda)) = \oplus_{\mu} m_{\mu;\Lambda} V_{\mu}$$

where $m_{\mu;\Lambda} \in \mathbb{N}$ ... multiplicity
characterizes degrees of freedom on the space

1.10 Quantized coadjoint orbits embedded in $\mathbb{R}^D$

There is a canonical quantization for the above Poisson bracket on adjoint orbit with suitably quantized orbit.

Fact:
All finite-dimensional irreps $V$ of $G$ are given by highest weight representations, with dominant integral highest weight $\Lambda \in \mathfrak{g}_0^*$

Here $\mathfrak{g}_0 \subset \mathfrak{g}$ is the Cartan subalgebra, i.e. max subalgebra of mutually commuting (i.e. diagonal) generators.

This means that $V = V_\Lambda$ has a unique highest weight vector $|\Lambda\rangle \in V$ with

$$X_i^+ |\Lambda\rangle = 0, \quad H |\Lambda\rangle = H[\Lambda] |\Lambda\rangle$$

for any (diagonal) Cartan generator $H$, and all other vectors in $V$ are obtained by acting repeatedly with lowering operators $X_i^-$ together with the Cartan generators.)

e.g. for $su(2)$: irreps characterized by spin, weights = eigenvale of $H = J_3$

Fact:
for compact Lie groups, there is a canonical isomorphism between the Lie algebra $\mathfrak{g}$ as a vector space and its dual space $\mathfrak{g}^*$, given by the standard Cartesian product $g_{ab} = \delta_{ab}$ on $\mathbb{R}^D$ (= Killing form).

In particular,

$$\Lambda \leftrightarrow H_\Lambda \quad (3)$$

Then coadjoint orbits $O(\Lambda)$ through $\Lambda$ are the same as adjoint orbits through $H_\Lambda$.

Given such a highest weight irrep $V_{N\Lambda}$, consider the matrix algebra

$$\mathcal{A}_N = End(V_{N\Lambda}) = Mat(N), \quad N = \dim V_{N\Lambda}$$
\( G \) acts naturally on \( \mathcal{A}_N \) via

\[
G \times \mathcal{A}_N \rightarrow \mathcal{A}_N
\]

\[
(g, M) \mapsto \pi(g)M\pi(g^{-1})
\]

(4)

where \( \pi \) ... rep. of \( G \) on \( V_{NA} \) → can decompose \( \mathcal{A} \) into harmonics = irreps:

\[
\mathcal{A}_N = \text{End}(V_{NA}) = V_{NA} \otimes V_{NA}^* = \bigoplus_{\mu} \tilde{m}_{NA;\mu} V_\mu
\]

\( \tilde{m}_{NA;\mu} \in \mathbb{N} \) ... multiplicity

can show:

\[
\tilde{m}_{NA;\mu} = m_{\lambda;\mu}
\]

for sufficiently large \( N \).

cf. (Hawkins q-alg/9708030, Pawelczyk & Steinacker hep-th/0203110)

moreover, can embed

\[
\mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1} \hookrightarrow \text{Pol}(\mathcal{O}(\Lambda))
\]

preserving the group action and norms.

Hence: \( \exists \) quantization map

\[
Q : \text{Pol}(\mathcal{O}(\Lambda)) \rightarrow \mathcal{A}_N
\]

(5)

\[
Y^\mu \mapsto \begin{cases} \hat{Y}^\mu_m & \mu < N \\ 0 & \mu \geq N \end{cases}
\]

(6)

(schematically)

which respects the group action, the norm and is one-to-one for modes with sufficiently small degree \( \mu \).

“correspondence principle”

in practice: rescale as desired

In particular: **monomials = Lie algebra generators**

\[
X^a := Q(x^a) = c_N \pi(\lambda_a) = X^a\dagger
\]

Their commutator reproduces Poisson bracket:

\[
[X^a, X^b] = ic_N f^{abc} X^c \xrightarrow{N \to \infty} 0
\]

(7)

\[
\{x^a, x^b\} = c_N f^{abc}x^c
\]

(8)

polynomial algebra generated by \( X^a \) generates full \( \mathcal{A}_N = \text{End}(V_{NA}) \).
Choose normalization e.g. such that

\[ X^a X^a = c_N^2 \pi(\lambda_a \lambda^a) \stackrel{1}{=} R^2 \]

here

\[ \pi(\lambda_a \lambda^a) = C^2 [N \Lambda] = (N \Lambda, N \Lambda + 2 \rho) \sim N^2 \quad \text{...quadratic Casimir} \quad (9) \]

\[ c_N \sim \frac{R}{N} \quad (10) \]

realize harmonics \( \hat{Y}^\mu_\nu(X) \sim Y^\mu_\nu(x) \) e.g. as completely symmetric (traceless ...) polynomials of given degree.

Therefore:

**Theorem 1.1.** \( A_N = \text{End}(V_{N \Lambda}) \) provides a quantization \( O_N(\Lambda) \) of the coadjoint orbit \( O(\Lambda) \), viewed as Poisson (symplectic) manifold embedded in \( \mathbb{R}^D \) with Poisson structure (8).

*same d.o.f. at low energies, but intrinsic UV cutoff.*

The quantized embedding map is given by

\[ X^a \propto \pi(\lambda^a) \]

The symplectic or Poisson structure is quantized such that

\[ (2\pi)^n \text{Tr} 1 = \int \frac{\omega^n}{n!} \]

where \( n = \text{dim } O(\Lambda) \)

**1.10.1 Example: fuzzy \( \mathbb{C}P^2 \)**

(Grosse & Strohmaier, Balachandran etal)

recall classical \( \mathbb{C}P^2 \):

\[ \mathbb{C}P^2 = \{ \lambda = g^{-1} \lambda_8 g, \quad g \in SU(3) \} \subset su(3) \cong \mathbb{R}^8 \quad \text{... (co)adjoint orbit} \]

\( \lambda = x^a \lambda_a \) satisfies embedding

\[ \delta_{ab} x^a x^b = 3, \quad d^{abc} x^a x^b = x^c. \quad (11) \]

harmonic analysis:
\[ \mathcal{C}(\mathbb{C}P^2) \cong \bigoplus_{k=1}^{\infty} (k,k) \]

**fuzzy version:**

\[ \mathcal{A}_N := \mathbb{C}P^2_N := \text{End}(V_N, \mathbb{C}) = \text{Mat}(d_N, \mathbb{C}) \cong \bigoplus_{k=1}^{N} (k,k) \]

\( V_N \) ... irrep of \( su(3) \) with highest weight \((0, N)\), \quad \( d_N = \text{dim} V_N = (N+1)(N+2)/2 \)

\[ X^a = c_N \pi_N (\lambda_a), \quad c_N = \frac{3}{\sqrt{N^2 + 3N}} , \]

is quantized embedding map

\[ X^a \sim x^a : \mathbb{C}P^2 \hookrightarrow \mathbb{R}^8 \]

can show: satisfies similar constraint

\[ [X_a, X_b] = \frac{i}{\sqrt{N^2 + 3N}} f_{abc} X_c , \quad (12) \]

\[ g_{ab} X_a X_b = 3, \quad (13) \]

\[ d_{abc} X_a X_b = \frac{N + \frac{3}{2}}{\sqrt{N^2 + 3N}} X_c \quad (14) \]

reduces to (11) for \( N \to \infty \),

Alexanian, Balachandran, Immirzi and Ydri hep-th/0103023, Grosse & Steinacker hep-th/0407089

### 1.11 Laplace operator on fuzzy \( \mathcal{O}_N(X) \):

Let \( \phi \in \mathcal{A}_N \) ... function on fuzzy \( \mathcal{O}_N(X) \)

\begin{definition}
\[ \square \phi := g_{ab} [X^a, [X^b, \phi]] \]
\end{definition}

where \( X^a = \pi(\lambda_a) = X^a \) ... quantized embedding operators (possibly rescaled). Recall that \( g \) acts via adjoint \( J_a \phi := i[X_a, \phi] \) on \( \mathcal{A}_N \)
hence
\[ \Box \phi = J_a J^a \phi \]
\[ \Box \hat{Y}_m = C^2[\mu]\hat{Y}_m \]

quadratic Casimir has same spectrum as classical Laplacian,
\[ \Box Y_m \propto C^2[\mu]Y_m \]

Thus \( \Box \) has the same spectrum on \( \mathcal{A}_N \) as \( \Box_g \) on \( C^\infty(\mathcal{O}(\Lambda)) \), up to cutoff.
hence:
\[ \Rightarrow \mathcal{O}_N(\Lambda) \text{ has the same effective (spectral) geometry as } \mathcal{O}(\Lambda). \]
This is much more general, as we will see.

2 Generic fuzzy spaces

Framework is not restricted to homogeneous spaces.
General setup: \( D \) hermitian matrices \( X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D \) describe quantized embedded symplectic space \((\mathcal{M}, \omega)\)

\[ \text{inherits pull-back metric (geometry), (quantized) Poisson / symplectic structure is encoded via } [X^\mu, X^\nu] = i\theta^{\mu\nu} \]

Define matrix Laplace operator on \( \mathcal{M}_N \) by
\[ \Box \phi := g_{ab}[X^a, [X^b, \phi]] \]
acting on \( End(\mathcal{H}) \)
Similarly, let \( \gamma_a, a = 1, ..., D \)...Gamma matrices associated to \( SO(D) \) acting on spinors \( V \)
\[ \{\gamma_a, \gamma_b\} = 2g_{ab} \]
Define matrix Dirac operator by
\[ \mathcal{D} := \gamma_a \otimes [X^a, .].\]
acting on $V \otimes \text{End}(\mathcal{H})$.
Arises naturally in matrix models. Its square is given by
\[ \mathcal{D}^2 = \Box + \Sigma^{ab}[X^a, X^b] \]
where $\Sigma^{ab} := \frac{1}{4}[\gamma^a, \gamma^b]$.
(cf. Lichnerowicz formula)

**Exercise 7:** check this relation.
These operators define a (spectral) geometry for $\mathcal{M}_N$.

### 2.1 Effective geometry of NC brane

consider scalar field moving on a fuzzy space, governed by “free” action
\[ S[\varphi] = -\text{Tr} \, [X^a, \varphi][X^b, \varphi] \, g_{ab} \]
\[ \sim \int \sqrt{|\theta^{-1}_{\mu\nu}|} \theta^{\mu\nu} \partial_{\mu} x^a \partial_{\nu} \varphi \theta^{\nu\rho} \partial_{\rho} x^b \partial_{\nu} \varphi \, g_{ab} \]
\[ = \int \sqrt{|G_{\mu\nu}|} G^{\mu\nu}(x) \partial_{\mu} \varphi \partial_{\nu} \varphi \]  \hspace{1cm} (15)

using $[f, \varphi] \sim i \theta^{\mu\nu}(x) \partial_{\mu} f \partial_{\nu} \varphi$
(assume dim $\mathcal{M} = 4$)

\[
\begin{align*}
G^{\mu\nu}(x) &= e^{-\sigma} \theta^{\mu\nu}(x) \theta^{\rho\sigma}(x) \, g_{\rho\sigma}(x) \quad \text{effective metric} \\
g_{\mu\nu}(x) &= \partial_{\mu} x^a \partial_{\nu} x^b g_{ab} \quad \text{induced metric on } \mathcal{M}
\end{align*}
\]

$\varphi$ couples to metric $G^{\mu\nu}(x)$, determined by $\theta^{\mu\nu}(x)$ & embedding
... quantized Poisson manifold with metric $(\mathcal{M}, \theta^{\mu\nu}(x), G_{\mu\nu}(x))$

**Exercise 8:** derive (15) with the above metric $G^{\mu\nu}$

### 2.1.1 The matrix Laplace operator

semi-classical limit of above matrix Laplacian:

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Theorem 2.1. \((M, \omega)\) symplectic manifold with \(\dim M \neq 2\), with \(x^a : M \hookrightarrow \mathbb{R}^D\) ... embedding in \(\mathbb{R}^D\) induced metric \(g_{\mu\nu}\) and \(G^{\mu\nu}\) as above. Then:

\[
\{x^a, \{x^b, \varphi\}\} g_{ab} = e^\sigma \Box_G \varphi
\]

\[
\Box_G = \frac{1}{\sqrt{G}} \partial_{\mu} (\sqrt{G} G^{\mu\nu} \partial_{\nu} \phi) \quad \text{Laplace-Op. w.r.t. } G_{\mu\nu}
\]

(H.S., [arXiv:1003.4134])

Hence:

\[
\Box \phi \sim -e^\sigma \Box_G \phi(x)
\]

For coadjoint orbits: \(G \sim g\) by group invariance, and \(\Box \sim \Box_g\) follows.

2.2 A degenerate fuzzy space: Fuzzy \(S^4\)

H. Grosse, C. Klimcik and P. Presnajder, hep-th/9602115

(sketch; for more details see e.g. Castelino, Lee & Taylor hep-th/9712105 or H.S. arXiv:1510.05779 )

Classical construction:

Consider fundamental representation \(\mathbb{C}^4\) of \(SU(4)\). Acting on a reference point \(z^{(0)} = (1, 0, 0, 0) \in \mathbb{C}^4\), \(SU(4)\) sweeps out the 7-sphere \(S^7 \subset \mathbb{R}^8 \cong \mathbb{C}^4\) → Hopf map

\[
S^7 \rightarrow S^4 \subset \mathbb{R}^5
\]

\[
z^a \mapsto x_i = z^a (\gamma_i)^a_{\beta} z^\beta \equiv \langle z|\gamma_i|z\rangle = tr(P_z \gamma_i), \quad P_z = |z\rangle\langle z|
\]

where \(\gamma_i\) are the \(\text{so}(5)\) gamma matrices.

Hence \(S^7\) is a bundle over \(S^3\) with fiber \(S^2\).

Recall \(\mathbb{C}P^3 = S^7/U(1)\). Can quantize this! →

Fuzzy construction:

Recall: \(\text{su}(4) \cong \text{so}(6)\) generated by \(\lambda^{ab} \in \text{so}(6)\)

Start with fuzzy \(\mathbb{C}P^3 \subset \mathbb{R}^{15} \cong \text{su}(4)\), generated by

\[
\mathcal{M}^{ab} = \pi_H (\lambda^{ab})
\]

acting on \(\mathcal{H}_N = (0, 0, N)\), for \(1 \leq a < b \leq 6\)

Hopf map corresponds to composition

\[
x^i : \mathbb{C}P^3 \rightarrow \mathbb{R}^{15} \overset{H}\rightarrow \mathbb{R}^5
\]
where $\Pi$ is projection of $\mathfrak{so}(6)$ to subspace spanned by $\lambda^6$, $i = 1, \ldots, 5$

in other words: Fuzzy $S^4_N$ is generated by

$$X^i := \mathcal{M}^i \mathcal{H} \quad \text{End}(\mathcal{H})$$

(18)

for $\mathcal{H} = (0, 0, N)$ satisfy

$$\sum_{a=1}^{5} X^a X^a = \frac{1}{4} N(N + 4) \mathbf{1}$$

$$[X_i, X_j] = i \mathcal{M}_{ij}$$

$$[\mathcal{M}_{ij}, X_k] = i(\delta_{ik} X_j - \delta_{jk} X_i)$$

(19)

Is fully $SO(5)$-covariant fuzzy space, since $\mathcal{M}_{ij}$, $i, j = 1, \ldots, 5$ generate $\mathfrak{so}(5)$.

Snyder-type fuzzy space!

Can see: local fiber is fuzzy $S^2_N$.

2.3 A self-intersecting fuzzy space: squashed $\mathbb{C}P^2$


classical construction:

Recall coadjoint orbit $\mathbb{C}P^2 \subset \mathbb{R}^8 \cong \mathfrak{su}(3)$

Consider projection map

$$\Pi : \mathbb{R}^8 \cong \mathfrak{su}(3) \rightarrow \mathbb{R}^6$$

projecting along the (simultaneously diagonalizable) Cartan generators $\lambda^3, \lambda^8$.

Then

$$x^a : \mathbb{C}P^2 \rightarrow \mathbb{R}^8 \rightarrow \mathbb{R}^6$$

sefines a 4-dimensional subvariety of $\mathbb{R}^6$ with a triple self-intersection at the origin

Fuzzy construction:

generators

$$X^a = \pi_{\mathcal{H}}(\lambda^a), a = 1, 2, 4, 5, 6, 7 \quad \in \text{End}(\mathcal{H})$$

acting on $\mathcal{H} = (0, N)$ generate fuzzy squashed $\Pi\mathbb{C}P^2_N$

arises as fuzzy extra dimensions in $\mathcal{N} = 4$ SYM with soft SUSY breaking potential, and in an analogous modified IKKT matrix model

(3 generations etc.)
2.4 lessons

- algebra $\mathcal{A} = End(\mathcal{H})$ ... quantized algebra of functions on $(M, \omega)$
  no geometrical information (not even $\text{dim}$)
  $\text{dim}(\mathcal{H})$ = number of “quantum cells”, $(2\pi)^n \text{Tr} Q(f) \sim \text{Vol}_\omega M$

- Poisson/symplectic structure encoded in C.R.

- every non-deg. fuzzy space locally $\approx \mathbb{R}^{2n}$ (cf. Darboux theorem!)

- geometrical info encoded in specific matrices $X^a$, $a = 1, ..., D$:

  $X^a \sim x^a : M \hookrightarrow \mathbb{R}^D$ ...embedding functions

  contained e.g. in matrix Laplacian $\Box = [X^a, [X^b, .]] \delta_{ab}$

  (or in coherent states, see below).
3 Lecture II: Applications: NC field theory & matrix models

goal: formulate physical models on fuzzy spaces
scalar field theory, gauge theory, (“emergent”) gravity
issues:
quantization ⇒ UV/IR mixing due to \( \Delta x^\mu \Delta x^\nu \geq L_{NC}^2 \)
→ strong non-locality, can be traced to string states
→ selects well-behaved model: IKKT model = maximal SUSY matrix model

(Yang-Mills) Matrix Models, IKKT model
• describes dynamical fuzzy branes = submanifolds \( \mathcal{M} \hookrightarrow \mathbb{R}^{10} \)
  interpreted as physical space-time
• NC gauge theory, dynamical geometry & emergent gravity
• closely related to string theory, introduced as non-perturbative description of string theory on \( \mathbb{R}^{10} \)
• well-behaved under quantization, due to maximal SUSY

3.1 Scalar field theory on \( S^2_N \)

consider \( A_N = \text{Mat}(N, \mathbb{C}) \) \( \hookrightarrow \) (Hilbert) space of functions on \( S^2_N \)

action for free real scalar field \( \phi = \phi^\dagger \):
\[
S_0[\phi] = \frac{4\pi}{N} \text{Tr}(\frac{1}{2} \phi \Box \phi + \frac{1}{2} \mu^2 \phi^2) \\
\sim \int_{S^2} (\frac{1}{2\pi N} \phi \Delta_g \phi + \frac{1}{2} \mu^2 \phi^2)
\]

harmonic (“Fourier”) decomposition
\[
\phi = \sum_{lm} \phi_{l,m} \mathbf{Y}_m^l, \quad \phi_{l,m}^\dagger = \phi_{l,-m}
\]
(finite!)
\[
S_0[\phi] = \frac{4\pi}{N} \sum_{l,m} (\phi_{l,m}(l(l+1) + \mu^2)\phi_{l,m})
\]

interacting real scalar field:
\[
S[\phi] = \frac{4\pi}{N} \text{Tr}(\frac{1}{2} \phi \Box \phi + \frac{1}{2} \mu^2 \phi^2 + \lambda \phi^4) \\
= \sum \phi_{l,m} \phi_{l,-m}(l(l+1) + \mu^2) + \lambda \sum \phi_{l_1,m_1} \cdots \phi_{l_n,m_n} V_{l_1,l_2;m_1,m_2} \cdots V_{l_n,l_4;m_1,m_4} \\
\]
\[
V_{l_1,l_2;m_1,m_2} = \text{Tr}(\mathbf{Y}_m^l \cdots \mathbf{Y}_m^l)
\]

... deformation of classical FT on \( S^2 \), built-in UV cutoff
3.2 scalar QFT on $S^2_N$

Feynman ”path“ (matrix) integral approach

$$Z[J] = \int \mathcal{D}\phi e^{-S[\phi]+\text{Tr}\phi J}$$

$$\langle \phi_{l_1m_1} \cdots \phi_{l_nm_n} \rangle = \frac{\int [\mathcal{D}\phi] e^{-S[\phi]} \phi_{l_1m_1} \cdots \phi_{l_nm_n}}{\int [\mathcal{D}\phi] e^{-S[\phi]}}$$

$$= \frac{1}{Z[0]} \frac{\partial^n}{\partial J_{l_1} \cdots \partial J_n} Z[J]|_{J=0}, \quad [\mathcal{D}\phi] = \mathcal{P} \mathcal{D}\phi_{lm}$$

... deformation & regularization of (euclid.) QFT on $S^2$, finite version of path integral,

UV cutoff

free QFT:

$$S_0[\phi] := \text{Tr} \frac{1}{2} \phi D\phi = \frac{1}{2} \sum_{l,m} \phi_{l-1,m-1} (l(l+1) + \mu^2) \phi_{l,m}$$  \hspace{1cm} (20)

Gaussian integral,

$$Z[J] = \int d\phi e^{-\text{Tr}(\frac{1}{2} \phi D\phi - \phi J)} = \int d\phi e^{-\text{Tr}(\frac{1}{2}(\phi - D^{-1}J)^\dagger D(\phi - D^{-1}J) + \frac{1}{2} \text{Tr} JD^{-1}J + \frac{1}{2} \text{Tr} J D^{-1}J)}$$

$$= \frac{1}{N} e^{\frac{1}{4} \text{Tr} J D^{-1}J}$$  \hspace{1cm} (21)

propagator:

$$\langle \phi_{l_1m_1} \phi_{l_2m_2} \rangle = \frac{1}{2} \int \prod d\phi_{lm} \phi_{l_1m_1} \phi_{l_2m_2} e^{-\sum_{l,m} \phi_{l-1,m-1} (l(l+1) + \mu^2)}$$

$$= \frac{1}{Z[0]} \frac{\partial^2}{\partial J_{l_1} \partial J_{l_2}} Z[J]|_{J=0}$$

$$= \delta_{l_1l_2} \delta_{m_1,-m_2} \frac{1}{l(l+1) + \mu^2}$$

as in commutative case, up to cutoff

$\rightarrow$ free field theory coincides with undeformed one.

interacting QFT:

$$Z[J] = \int \mathcal{D}\phi e^{-S_0[\phi] - S_{\text{int}}[\phi] + \text{Tr}\phi J}$$

$$= e^{-S_{\text{int}}[\partial \phi]} \int \mathcal{D}\phi e^{-S_0[\phi] + \text{Tr}\phi J}$$

$$= e^{-S_{\text{int}}[\partial \phi]} Z[J]$$

$$\langle \phi_{l_1} \cdots \phi_{l_n} \rangle = \frac{1}{Z[0]} \frac{\partial^n}{\partial J_{l_1} \cdots \partial J_n} e^{-S_{\text{int}}[\partial \phi]} Z[J]|_{J=0}$$
perturbative expansion \( \Rightarrow \) Wick’s theorem,

\[
\langle \phi_{I_1} \ldots \phi_{I_{2n}} \rangle = \sum \text{contractions} \langle \phi \phi \rangle \ldots \langle \phi \phi \rangle
\]

vertices: e.g.

\[
S_{\text{int}} = \frac{1}{\pi} Tr \phi^4 = Tr V,
\]

\[
V = \lambda \sum \phi_{l_1 m_1} \cdots \phi_{l_n m_n} V_{l_1 \ldots l_4, m_1 \ldots m_4}
\]

finite, but distinction planar \( \leftrightarrow \) nonplanar diagrams

results: hep-th/0106205

- large phase factors, interaction vertices rapidly oscillating for \( \frac{H \Lambda^2}{E} \geq \Lambda_{NC}^2 \)
  (loop effects probe area quantum \( \Delta A \sim 1/N \))
- 1-loop effective action

\[
S_{\text{one-loop}} = S_0 + \int \frac{1}{2} \Phi \left( \delta \mu^2 - \frac{g}{12\pi} h(\Delta) \right) \Phi + o(1/N)
\]

with

\[
h(l) = -\frac{1}{2} \int_{-1}^{1} dt \frac{1}{1-t} (P(t)-1) = \left( \sum_{k=1}^{l} \frac{1}{k} \right), \quad \delta \mu^2 = \frac{g}{8\pi} \sum_{j=0}^{N} \frac{2j + 1}{j(j+1) + \mu^2}
\]

Chu Madore HS [hep-th/0106205]

does NOT agree with usual QFT on \( S^2 \),

"anomalous contributions" to quantum effective action

(=finite version of UV/IR mixing)

central feature of NC QFT, obstacle for perturb. renormalization

Minwalla, V. Raamsdonk, Seiberg hep-th/9912072]

- new physics, non-local
4 Matrix models and NC gauge theory

4.1 Matrix model for $S^2_N$

\[
S[X] = \frac{1}{g^2} \text{Tr} \left( [X^a, X^b] [X_a, X_b] - 4i \varepsilon_{abc} X^a X^b X^c - 2X^a X_a \right)
\]
\[
= \frac{1}{g^2} \text{Tr} \left( [X^a, X^b] - i \varepsilon_{abc} X_c \right) \left( [X_a, X_b] - i \varepsilon_{abc} X^c \right)
\]
\[
= \frac{1}{g^2} \text{Tr} F_{ab} F_{ab} \geq 0
\]
where $X^a \in \text{Mat}(N, \mathbb{C})$, $a = 1, 2, 3$ and 

\[
F_{ab} := [X^a, X^b] - i \varepsilon^{abc} X_c 
\]

field strength solutions (minima):

\[
F_{ab} = 0 \iff [X^a, X^b] = i \varepsilon^{abc} X_c
\]

$X^a = \lambda^a$, $\lambda^a$ ... rep. of $su(2)$

any rep. of $su(2)$ is a solution! $X^a = \begin{pmatrix} \lambda^a_{(M_1)} & 0 & \ldots & 0 \\ 0 & \lambda^a_{(M_2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda^a_{(M_k)} \end{pmatrix}$

concentric fuzzy spheres $S^2_{M_i}$!

geometry & topology dynamical!

expand around solution:

\[
X^a = \lambda^a + A^a \in \text{Mat}(N, \mathbb{C})
\]

\[
F_{ab} = [\lambda^a, A^b] - [\lambda^b, A^a] - i \varepsilon^{abc} A_c + [A^a, A^b] (\sim \text{"d}A + A A^n)
\]

can be interpreted in terms of

\[
\begin{cases}
U(1) \text{ gauge theory on } S^2_N \ (\text{tang. fluct. if}) \ 
\lambda^a A_a = 0 \\
\text{coupled to scalar field } D_{\mu} \phi D^{\mu} \phi \ (\text{radial fluctuations}) \ X^a = \lambda^a (1 + \phi)
\end{cases}
\]

however:

radial deformation = deformation of embedding, geometry!

\[
geometry \leftrightarrow \text{NC gauge theory} ?!
\]

above matrix model describes dynamical fuzzy space
4.2 Gauge theory on \( \mathbb{R}^4_\theta \)

Let \( [\bar{X}^\mu, \bar{X}^\nu] = i\bar{\theta}^{\mu\nu}, \quad \bar{X}^\mu \in \mathcal{A} = \mathcal{L}(\mathcal{H}) \) (Moyal-Weyl) consider fluctuations around \( \mathbb{R}^4_\theta \):

\[
X^\mu = \bar{X}^\mu - \bar{\theta}^{\mu\nu} A_\nu, \quad A_\nu \in \mathcal{A}
\]

Define derivative operator on \( \mathbb{R}^4_\theta \) by

\[
[X^\mu, \phi] = i\bar{\theta}^{\mu\nu} \partial_\nu \phi \rightarrow \]

\[
[X^\mu, X^\nu] = i\bar{\theta}^{\mu\nu} - i\bar{\theta}^{\mu\nu'} (\partial_\nu' A_{\nu'} - \partial_{\nu'} A_{\nu} + i[A_{\mu'}, A_{\nu'}]) = i\bar{\theta}^{\mu\nu} - i\bar{\theta}^{\mu\nu'} F_{\mu'\nu'}
\]

\[ F_{\mu\nu}(x) \ldots \text{u}(1) \text{ field strength} \]

Exercise 12: check this formula (and the gauge transformation law below)

Yang-Mills action:

\[
S_{YM}[X] = \text{Tr}[X^\mu, X^\nu][X^\mu', X^\nu'] \delta_{\mu\nu'} \delta_{\mu'\nu'} = \rho \int d^4x F_{\mu\nu} F_{\mu'\nu'} \bar{G}^{\mu\nu'} G^{\mu'\nu'} + \partial() \]

(up to surface term \( \text{Tr}[X, X] = \int F \rightarrow 0 \))

... NC \( \text{U}(1) \) gauge theory on \( \mathbb{R}^4_\theta \),

effective metric

\[
\bar{G}^{\mu\nu} = \bar{\theta}^{\mu\mu'} \bar{\theta}^{\nu\nu'} \delta_{\mu'\nu'}, \quad \rho = |\bar{\theta}^{-1}_{\mu\nu}|^{1/2}
\]

gauge transformations:

\[
X^\mu \rightarrow U X^\mu U^{-1} = U(\bar{X}^\mu - \bar{\theta}^{\mu\nu} A_\nu)U^{-1} = \bar{X}^\mu + U[\bar{X}^\mu, U^{-1}] - \bar{\theta}^{\mu\nu} U A_\nu U^{-1} \]

\[ = \bar{X}^\mu - \bar{\theta}^{\mu\nu} (U \partial_\nu U^{-1} + U A_\nu U^{-1}) \]

infinitesimal \( U = e^{i\Lambda(X)}, \quad \delta A_\mu = i\partial_\mu \Lambda(X) + i[\Lambda(X), A_\mu] \)

invariant under gauge trafo

\[
F_{\mu\nu} \rightarrow U F_{\mu\nu} U^{-1} \sim \text{symplectomorphism} \]

Yang-Mills matrix model \( S_{YM} \) describes \( \text{U}(1) \) gauge theory on \( \mathbb{R}^4_\theta \)
no “local” observables! (need trace)

coupling to scalar fields:

consider

$$S[X, \phi] = -\operatorname{Tr} \left( [X^\mu, X^\nu] [X^\mu', X^\nu'] \delta_{\mu\nu} \delta_{\mu'\nu'} + [X^\mu, \phi] [X^\mu', \phi] \delta_{\mu\mu'} \right)$$

$$= \rho \int d^4x \left( F_{\mu\nu} F_{\mu'\nu'} \bar{G}^{\mu\nu} \bar{G}^{\mu'\nu'} + D_\mu \phi D_\nu \phi \bar{G}^{\mu\nu} \right)$$

$$[X^\mu, \phi] = i \bar{\theta}^{\mu\nu} (\partial_\nu + i [A_\mu, \cdot]) \phi =: i \bar{\theta}^{\mu\nu} D_\mu \phi$$

gauge transformation

$$\phi \to U \phi U^{-1}$$ (adjoint)

same form as

$$S[X] = \operatorname{Tr} [X^a, X^b] [X^a', X^b'] \delta_{ab} \delta_{ab'}$$, \quad a = 1, ..., 4+1

more generally: \( D = 10 \) matrix model around \( \mathbb{R}_{\theta}^4 \):

$$S[X] = -\operatorname{Tr} [X^a, X^b] [X^a', X^b'] \delta_{ab} \delta_{ab'} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}_{\theta}^4} d^4x \rho \left( G^{\mu\nu} G^{\mu'\nu'} F_{\mu\nu} F_{\mu'\nu'} + G^{\mu\nu} \eta_{\mu\nu} \right)$$

$$+ 2 \bar{G}^{\mu\nu} D_\mu \phi^i D_\nu \phi^j \delta_{ij} + [\phi^i, \phi^j] \delta_{ij}$$

... same as bosonic part of \( \mathcal{N} = 4 \) SYM!

generalization to \( U(n) \):

new background

$$X^a \otimes 1_n$$

naturally interpreted as \( n \) coincident branes. fluctuations

$$\begin{pmatrix} X^\mu \\ \phi^i \end{pmatrix} = \begin{pmatrix} \bar{X}^\mu & 0 \\ 0 & 1_n \end{pmatrix} + \begin{pmatrix} A^\mu \\ \phi^i \end{pmatrix}$$,

it is easy to see that

$$A^\mu = -\theta^{\mu\nu} A_{\nu\alpha}(\bar{X}) \lambda^\alpha$$,

$$\phi^i = \phi^i_{\alpha}(\bar{X}) \lambda^\alpha$$

... \( u(n) \)-valued gauge resp. scalar fields on \( \mathbb{R}_{\theta}^4 \), denoting with \( \lambda^\alpha \) a basis of \( u(n) \).

The matrix model

$$S = -\operatorname{Tr} [X^a, X^b] [X^a', X^b'] \delta_{ab} \delta_{ab'} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}_{\theta}^4} d^4x \rho \left( G^{\mu\nu} G^{\mu'\nu'} F_{\mu\nu} F_{\mu'\nu'} + G^{\mu\nu} \eta_{\mu\nu} 1_n \right)$$

$$+ 2 \bar{G}^{\mu\nu} D_\mu \phi^i D_\nu \phi^j \delta_{ij} + [\phi^i, \phi^j] \delta_{ij}$$

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where \( tr() \) ... trace over the \( u(n) \) matrices,
\( F_{\mu\nu} \) ... \( u(n) \) field strength.
... \( u(n) \) Yang-Mills on \( \mathbb{R}^4_0 \)

note:

- extremely simple mechanism:
  gauge fields = fluctuations of dynamical matrices  
  \[ X^\mu \rightarrow X^\mu + A^\mu \]
  “covariant coordinates”
  works only on NC spaces!

- matrix models \( \text{Tr}[X, X][X, X] \sim \) gauge-invariant YM action

- generalized easily to \( U(n) \) theories but  
  \( U(1) \) sector does not decouple from \( SU(n) \) sector

- one-loop: UV/IR mixing \( \rightarrow \) not QED, problem  
  except in \( \mathcal{N} = 4 \) SUSY case: finite (!?)

- closer inspection: \( U(1) \) sector is part of geometric sector,  
  \( \rightarrow \) emergent “gravity“.