Closed star product on noncommutative $\mathbb{R}^3$ and scalar field dynamics

Quantum Structure of Spacetime and Gravity, Belgrade

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Physical motivation for NCG

- DFR, Commun. Math. Phys. 172: 187-220 (1995)

  „...**A sufficient condition for preventing gravitational collapse can be expressed as an uncertainty relation for the coordinates. This relation can in turn be derived from a commutation relation for the coordinates.**"

$$\Delta x_\mu \Delta x_\nu > l_{\text{Planck}}^2$$

$$x_\mu \to \hat{x}_\mu \Rightarrow [\hat{x}_\mu, \hat{x}_\nu] \neq 0$$

- Certain low energy limits of string theory (Moyal space) and Loop quantum gravity (κ-Minkowski) lead to NCFT
- **Not only for Planck scale physics** $\longrightarrow$ Almost-commutative manifolds: reformulation of gauge theories and the "mathematical" origin of Higgs mechanism and Standard model (Dubois-Violette, Kerner, Madore, Connes,...)

NCFT - modern history

- 1986- Witten: string field theory
- 1990- fuzzy sphere, κ-Minkowski
- 1992- Yang-Mills-Higgs model from matrix geometry
- 1998- NCFT on $\mathbb{R}^4_\theta$ as some low energy limit of string
- $>$1998 more interests: renormalizability, matrix model formulation, NC gauge theories...
- 2004- exactly solvable and to all order renormalizable model on $\mathbb{R}^4_\theta$
- $>$2004 $\uparrow\uparrow$ literature on NCFT on $\mathbb{R}^4_\theta$, κ-Minkowski, $\mathbb{R}^3_\lambda$, NC Tori, etc.

NCFT - properties

- nonlocal theories with complicated kinetic operator
- some could be represented as matrix models
- UV/IR mixing
- vacuum instabilities

Scalar field theory on deformed $\mathbb{R}^3$

Scalar field theory on deformed $\mathbb{R}^3$

T.J., T. Poulain and J.C. Wallet, "*Closed star product on noncommutative* $\mathbb{R}^3$ *and scalar field dynamics*," arXiv:1603.09122 [hep-th].

Properties:

- UV/IR freedom
- one-loop finite 2-point function $\longrightarrow$ finite n-point functions
- fulfilling the long forgotten dream of noncommutativity serving as a natural UV cut-off

Deformation of $\mathbb{R}^3$ space $\longrightarrow \mathbb{R}^3_\theta$ generated by $\hat{x}_i$ satisfying

$$[\hat{x}_i, \hat{x}_i] = i\theta\epsilon_{ijk}\hat{x}_k$$

It will be convenient to view the algebra $\mathbb{R}^3_\theta$ as

$$\mathbb{R}^3_\theta := (\mathcal{M}(\mathbb{R}^3), \star_{\mathcal{D}}),$$

where $\mathcal{M}(\mathbb{R}^3)$ is the multiplier algebra of $\mathcal{S}(\mathbb{R}^3)$ (the set of Schwartz functions on $\mathbb{R}^3$) for the star-product $\star_{\mathcal{D}}$ defined by

$$f \star_{\mathcal{D}} g = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\widetilde{f}(k_1)\widetilde{g}(k_2)\frac{2|B(k_1,k_2)|\sin(\frac{\theta}{2}|k_1|)\sin(\frac{\theta}{2}|k_2|)}{\theta|k_1||k_2|\sin\left(\frac{\theta}{2}|B(k_1,k_2)|\right)}e^{iB_\mu(k_1,k_2)x^\mu}$$

for any $f, g \in \mathcal{S}(\mathbb{R}^3)$ in which the symbol $\widetilde{f}$ denotes generically the Fourier transform of f.

$\star_{\mathcal{D}}$ is closed under the trace functional

$$\int f \star_{\mathcal{D}} g = \int f \;\; g$$

which will enable us to have a free field theory with a commutative Laplacian

$$S = \int d^3x \big[\frac{1}{2}\partial_\mu\phi\star_{\mathcal{D}}\partial_\mu\phi + \frac{1}{2}m^2\phi\star_{\mathcal{D}}\phi\big] = \int d^3x \big[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi\phi\big]$$

as shown in V.G. Kupriyanov and P. Vitale, "*Noncommutative $\mathbb{R}^d$ via closed star product*", JHEP 08 (2015) 024, [arXiv:1502.06544]

Interaction

The model is defined with the following interaction

$$\begin{aligned} S_{int} &= \lambda \int d^3 x \phi \star_{\mathcal{D}} \phi \star_{\mathcal{D}} \phi \star_{\mathcal{D}} \phi \\ &= \lambda \int d^3 x \int \left[\prod_{i=1}^{4} \frac{d^3 k_i}{(2\pi)^3} \tilde{\phi}(k_i) \right] (e^{ik_1 x} \star_{\mathcal{D}} e^{ik_2 x} \star_{\mathcal{D}} e^{ik_3 x} \star_{\mathcal{D}} e^{ik_4 x})(x) \\ &= \lambda \int \left[\prod_{i=1}^{4} \frac{d^3 k_i}{(2\pi)^3} \tilde{\phi}(k_i) \right] \mathcal{W}(k_1, k_2) \mathcal{W}(k_3, k_4) \delta(B(k_1, k_2) + B(k_3, k_4)) \end{aligned}$$

Two-point functions

For $n \geq 4$, n-point functions are already finite in the commutative case for $\mathbb{R}^3$.
We are interested in the 2-point function $\longrightarrow$ two type of contributions (two contractions).

$$\Gamma_2^{(I)} = \int d^3x \left[\prod_{i=1}^{4} \frac{d^3k_i}{(2\pi)^3}\right] \widetilde{\phi}(k_3)\widetilde{\phi}(k_4)\frac{\delta(k_1+k_2)}{k_1^2+m^2}(e^{ik_1x}\star_{\mathcal{D}}e^{ik_2x}\star_{\mathcal{D}}e^{ik_3x}\star_{\mathcal{D}}e^{ik_4x})(x)$$

$$\Gamma_2^{(II)} = \int d^3x \left[\prod_{i=1}^{4} \frac{d^3k_i}{(2\pi)^3}\right] \widetilde{\phi}(k_2)\widetilde{\phi}(k_4)\frac{\delta(k_1+k_3)}{k_1^2+m^2}(e^{ik_1x}\star_{\mathcal{D}}e^{ik_2x}\star_{\mathcal{D}}e^{ik_3x}\star_{\mathcal{D}}e^{ik_4x})(x)$$

Type-I contributions

Since

$$(e^{ikx} \star_{\mathcal{D}} e^{-ikx})(x) = \frac{4}{\theta^2} \frac{\sin^2(\frac{\theta}{2}|k|)}{|k|^2}$$

we obtain

$$\begin{aligned}
\Gamma_2^{(I)} &= \int d^3x \frac{d^3k_3}{(2\pi)^3} \frac{d^3k_4}{(2\pi)^3} \widetilde{\phi}(k_3)\widetilde{\phi}(k_4)(e^{ik_3x} \star_{\mathcal{D}} e^{ik_4x})(x)\omega^{(I)} \\
&= \int d^3x (\phi \star_{\mathcal{D}} \phi)(x)\omega^{(I)} = \int d^3x\ \phi(x)\phi(x)\omega^{(I)}
\end{aligned}$$

with

$$\omega^{(I)} = \frac{4}{\theta^2} \int \frac{d^3k}{(2\pi)^3} \frac{\sin^2(\frac{\theta}{2}|k|)}{k^2(k^2+m^2)} = \frac{1}{\pi^2\theta^2} \int_0^\infty dr \frac{1-\cos(\theta r)}{r^2+m^2} = \frac{1-e^{-\theta m}}{2m\pi\theta^2}$$

Type-I contributions

- Type-I contributions are UV finite and do not exhibit IR singularity.
- Whenever $\theta \neq 0$, type-I contributions cannot generate IR/UV mixing.

The θ expansion of $\omega^{(I)}$ gives

$$\omega^{(I)}_{\theta \to 0} = \Lambda + ...,$$

where the ellipsis denote finite ($\mathcal{O}(1)$) contributions and $\Lambda = \frac{1}{2\pi\theta}$. Thus, we recover as leading divergent term the expected linear divergence (showing up when $\Lambda \to \infty$) which occurs in the 2-point function for the commutative theory with $\Lambda = \frac{1}{2\pi\theta}$ as the UV cutoff.

Type-II contributions

$$\Gamma_2^{(II)} = \Gamma_2^{(I)} + \int \frac{d^3k_2}{(2\pi)^3}\frac{d^3k_4}{(2\pi)^3}\widetilde{\phi}(k_2)\widetilde{\phi}(k_4)I(k_2,k_4)$$

$$I(k_2,k_4) \sim \frac{C(\alpha_2,\beta_2)}{\theta}|k_2|u^n\delta_n^{'}(k_4) + \mathcal{O}(\theta^0),$$

with $C(\alpha_2,\beta_2)$ is finite. Hence, as for the type-I contributions, the θ expansion of $I(k_2,k_4)$ is

$$I \sim \Lambda + ... \tag{1}$$

where the ellipsis still denote finite $\mathcal{O}(1)$ contributions and $\Lambda = \frac{1}{2\pi\theta}$. Thus, we recover one more time the expected linear divergence when $\Lambda \to \infty$ ($\theta \to 0$) occurring in the 2-point function for the commutative theory. Again, the present $\mathfrak{su}(2)$ noncommutativity generates a natural UV cutoff for the scalar field theory.

Thank you for your attention!