Non-relativistic limit of κ-Minkowski Dirac equation

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Motivation

NC geometry $\rightarrow$ Quantum gravity
$\rightarrow$ motion of charged particle in $\vec{B} = non - const$ field

κ-Minkowski: Originaly invariant under κ-Poincare (quantum group symetry) [Lukierski et. al. '93]

NC spacetime with non-constant noncommutativity (non-constant $\vec{B}$)

Twist formalism enables us to construct NC gauge theories

Posible to construct cyclic integral
[Aschieri, Castellani, '09]

κ-Minkowski

Originally defined by commutation relations between coordinates:

$$[\hat{x}^0, \hat{x}^i] = ia\hat{x}^i, \qquad a = \frac{1}{\kappa}$$

$$[\hat{x}^j, \hat{x}^i] = 0$$

Our approach: deform Minkowski space-time by an Abelian twist to obtain commutation relations

Choice of twist is not unique (later...)

Abelian twist

Twist is used to deform a symmetry Hopf algebra
Twist $\mathcal{F}$ is invertible bidifferential operator from the universal enveloping algebra of the symmetry algebra

We work in 4D and deform the Minkowski space-time by the following twist

$$\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta_{ab}X^a \bigotimes X^b}$$
$$[X^a, X^b] = 0, \quad \text{a,b=1,2} \qquad X_1 = \partial_0 \text{ and } X_2 = x^i\partial_i$$
$$\mathcal{F} = \mathrm{e}^{\frac{-ia}{2}(\partial_0 \otimes x^j\partial_j - x^j\partial_j \otimes \partial_0)}$$

Bilinear maps are deformed by twist!
Bilinear map μ
$\mu : X \times Y \to Z$
$\mu_* = \mu\mathcal{F}^{-1}$

$\star$-Product

Example:

twist deformation of the usual multiplication is a product in NC space-times: the $\star$-product

$$f \star g = \mu_\star \{f \otimes g\} = \mu\{\mathcal{F}^{-1} f \otimes g\} = fg + \frac{ia}{2} x^j (\partial_0 f \partial_j g - \partial_j f \partial_0 g) + \ldots$$

It reproduces the commutation relations of κ-Minkowski

Twisted differential calculus

Vector fields act on a forms via Lie derivatives!
The exterior derivative is the classical one

$$df = (\partial_\mu f) dx^\mu = (\partial^\star_\mu f) \star dx^\mu; \quad \partial^\star_0 = \partial_0, \quad \partial^\star_j = \mathrm{e}^{-\frac{i}{2} a \partial_0} \partial_j$$
$$d^2 = 0$$
$$d(f \star g) = df \star g + f \star dg$$
$$f \star dx^0 = dx^0 \star f$$
$$f \star dx^j = dx^j \star \mathrm{e}^{ia\partial_0} f$$
$$dx^\mu \wedge_\star dx^\nu = dx^\mu \wedge dx^\nu, \quad dx^0 \wedge ... \wedge dx^3 = d^4x$$

The usual integral of maximal form is cyclic:
$\int \omega_1 \wedge_\star \omega_2 = (-1)^{p_1 p_2} \int \omega_2 \wedge_\star \omega_1 + surface$

Action for the spinor field ψ

$$S = \int (\bar{\psi} \star d\psi \wedge_\star V \wedge_\star V \wedge_\star V - m\bar{\psi} \star \psi \wedge_\star V \wedge_\star V \wedge_\star V \wedge_\star V)$$
$V = \gamma_\mu dx^\mu, \quad \gamma_\mu$ -standard 4D Dirac gamma matricies

$$S = \int (\bar{\psi} \star \gamma^\mu \partial^\star_\mu \psi - m\bar{\psi} \star \psi) \star d^4x = \int (\bar{\psi} \star \gamma^\mu \partial^\star_\mu \psi - m\bar{\psi} \star \psi) d^4x$$

Equation of motion in 1st order in a:
$i\gamma^\mu \partial_\mu \psi - m\psi + \frac{a}{2}\gamma^j \partial_0 \partial_j \psi = 0$ [Dimitrijević, Jonke, '11]
(Unlike in $\theta = const$, where the action for the free spinor field is undeformed!)

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Possible application in condensed matter physics

Non-relativistic limit [Bjorken, Drell]

$\psi = \begin{bmatrix} \varphi \\ \chi \end{bmatrix} \mathrm{e}^{-\mathrm{i}mt} \qquad \mathrm{i}\partial_0\chi, \mathrm{i}\partial_0\varphi \lll m\chi$

$\mathrm{i}\partial_0\varphi = \frac{p^2}{2}(\frac{1}{m} - a)\varphi = \frac{p^2}{2m_{eff}}$

Schrodinger equation with $m_{eff} = \frac{m}{1-am} > m$

Adding EM field, U(1) gauge theory [Dimitrijević, Jonke, '11] leads to:

$\mathrm{i}\partial_0\varphi = (\frac{D^j D_j}{2m_{eff}} - \frac{e\vec{\sigma}\vec{B}}{2m}(1 - 2ma))\varphi +$ other interactions

effective mass, effective gyromagnetic ratio $g_{eff} = 2(1 - 2ma)$

Future Work: a more realistic model

Request a symetry of the problem
3+1 dimensions, motion of an electron in the plane
magnetic field orthogonal to the plane $\vec{B} = B\vec{e_z}$

-translations along z-axis
-rotations around z-axis

Abelian twist given by $\mathcal{F} = \mathrm{e}^{-\frac{a}{2}(P_3 \otimes M_{12} - M_{12} \otimes P_3)}$
$M_{12} = i(\partial_1 x_2 - \partial_2 x_1) \quad P_3 = -i\partial_3$

Commutation relations between coordinates are not κ-Minkowski, but still Lie algebra type

$[x^0 ,^\star x^j] = 0 \qquad [x^1 ,^\star x^2] = 0$

$[x^3 ,^\star x^1] = -iax^2 \quad [x^3 ,^\star x^2] = iax^1$

The symmetry algebra is the twisted Poincare algebra
Diff. calculus, NC gauge theory, applications. . .

Thank you!!!