Quantum entanglement of Pais-Uhlenbeck oscillators

Stefan Mladenov

Sofia University

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Outline

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- The PU oscillator is conformally invariant for frequencies $\omega_k = (2k+1)\omega_0$.
- Stable coherent states, which have constant dispersions and a modified Heisenberg uncertainty relation.

An alternative Hamiltonian

The EoM of the PU oscillator of order $2n$ can be obtained by varying the action

$$S = \frac{1}{2} \int dt \, x_i \prod_{k=0}^{n-1} \left(\frac{d^2}{dt^2} + \omega_k^2 \right) x_i .$$

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The variables, which satisfy the structure relations $\{x_i^k, p_j^m\} = \delta_{ij}\delta_{km}$, are

$$x_i^k = \sqrt{|\alpha_k|\rho_k}\prod_{\substack{m=0\\ m\neq k}}^{n-1}\left(\frac{d^2}{dt^2}+\omega_m^2\right)x_i , \qquad p_i^k = \mathrm{sgn}(\alpha_k)\frac{dx_i^k}{dt} .$$

Set-up: ring of PU oscillators

$$H_N = \frac{1}{2}\sum_{\mu=1}^{N}\sum_{k=0}^{1} \mathrm{sgn}(\alpha_{\mu,k}) \left(p_\mu^k p_\mu^k + \omega_{\mu,k}^2 x_\mu^k x_\mu^k\right) + \frac{1}{2}\sum_{\langle\mu,\nu\rangle=1}^{N}\sum_{k,l=0}^{1} \tilde{c}_{\mu\nu}^{kl} x_\mu^k x_\nu^l$$

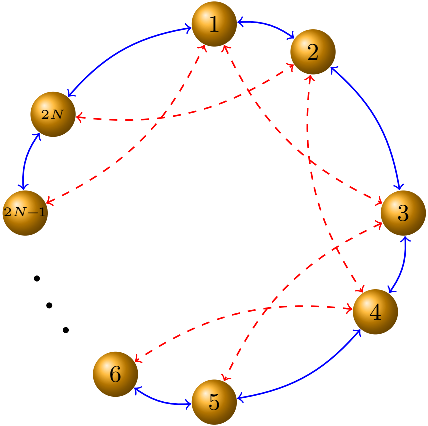

Figure: Closed chain of N identical PU oscillators.

Diagonalisation

The Hamiltonian can be written in matrix form as

$$H_N = \frac{1}{2}\eta^{\mathrm{T}} \begin{pmatrix} \Omega & 0 \\ 0 & \mathbb{1}_{2N} \end{pmatrix} \eta, \qquad \Omega = \begin{pmatrix} W & C & 0 & \cdots & C \\ C & W & C & \cdots & 0 \\ 0 & C & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ C & 0 & \cdots & C & W \end{pmatrix}.$$

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Symmetric block circulant matrices with symmetric blocks are diagonalised by Discrete Fourier Transform (DFT):

$$\widehat{\Omega} = U^{-1}\Omega U = \mathrm{diag}[D_1, D_2, \dots, D_N],$$
$$U_{kl} = \frac{1}{\sqrt{N}} e^{2\pi i kl/N}\mathbb{1}_2, \quad k, l = 0, 1, \dots, N-1,$$

where the eigenvalues of Ω are given by

$$D_{k+1} = W + 2\cos\left(2\pi k/N\right) C.$$

Building the Fock space

The creation and annihilation operators are defined by

$$\mathfrak{a}_j = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\lambda_j}\widehat{x}_j + \frac{i}{\sqrt{\lambda_j}}\widehat{p}_j\right), \qquad \mathfrak{a}_j^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\lambda_j}\widehat{x}_j - \frac{i}{\sqrt{\lambda_j}}\widehat{p}_j\right).$$

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The Fock space is built up from the vacuum

$$|\{n_j\}\rangle = \prod_{j=1}^{2N} \frac{(a_j^\dagger)^{n_j}}{\sqrt{n_j!}} |\{0\}\rangle, \qquad |\{n_j\}\rangle = |n_1\rangle \otimes \cdots \otimes |n_{2N}\rangle.$$

The excited states are orthonormal, $\langle\{m_j\}|\{n_j\}\rangle = \delta_{\{m_j\},\{n_j\}}$.

Thermo-field dynamics (TFD)

Having the Hamiltonian diagonalised, the standard density matrix is:

$$\rho_{\mathsf{eq}}(K_j) = \frac{1}{Z(K_j)} e^{-\beta H_N}, \qquad Z(K_j) := \mathrm{Tr}_{\{j\}} e^{-\beta H_N}.$$

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- Extended entanglement entropy:

$$\widehat{S}_{1,2} = -k_{\text{B}} \mathrm{Tr}_{1,2} \left[\widehat{\rho}_{1,2} \log \widehat{\rho}_{1,2}\right].$$

Entanglement entropy

$$\widehat{S}_{1,2}(K_1, K_2) = k_{\mathrm{B}} \left[\frac{K_1}{e^{K_1} - 1} + \frac{K_2}{e^{K_2} - 1} - \log \left[\left(e^{-K_1} - 1 \right) \left(e^{-K_2} - 1 \right) \right] \right]$$

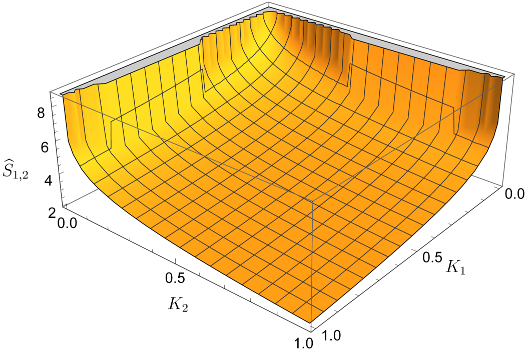

Figure: The entanglement entropy as function of K_1 and K_2 in units $k_{\mathrm{B}} = 1$.
Evidently, the Nernst heat theorem is satisfied.

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Fisher metric $g_{\mu\nu}(K_1, K_2) = \partial_\mu \partial_\nu S(K_1, K_2)$

$$g = \begin{pmatrix} \frac{k_B e^{K_1}\left[e^{K_1}(K_1-1)+K_1+1\right]}{\left(e^{K_1}-1\right)^3} & 0 \\ 0 & \frac{k_B e^{K_2}\left[e^{K_2}(K_2-1)+K_2+1\right]}{\left(e^{K_2}-1\right)^3} \end{pmatrix}$$

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Thank you for your attention!