PROBLEMS

SPECTRAL TRIPLES AND HEAT KERNEL EXPANSION

1. The Borsuk-Ulam Theorem reads as follows. For any $n \in \mathbb{N}$ let $\sigma : x \to -x$ be the antipodal involution of S^n. Then if $F : S^n \to \mathbb{R}^n$ is continuous, there exists $x \in S^n$ such that $F(x) = F(\sigma(x))$. Try to reformulate this Theorem in the NC geometry language (L. Dabrowski 1504.03588).

2. Prove that the set of states is convex. Start with $C(M)$.

3. Let $M = \{0, 1\}$ be a space consisting of two points. Then $C(M) = \mathbb{C}^2$.
   (a) What is (the minimal choice for) $\mathcal{H}$? What is $\pi(a)$?
   (b) What are the states?
   (c) What are pure states?
   (d) Take
   $$D = \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix}$$
   with $f \in \mathbb{R}$. Compute the distance function.

4. If $\Gamma(s)\,\zeta(s, L)$ has a double pole, what can you say about the $t \to 0$ asymptotic expansion of $K(t, L)$?

5. Prove that the heat kernel of $L = -\partial_\mu^2$ on $\mathbb{R}^d$ has the form
   $$K(x, y|t) = (4\pi t)^{-d/2} \exp\left(-\frac{(x-y)^2}{4t}\right)$$

6. Compute the numerical coefficients in front of ER^2 and $E\nabla^2 E$ in $a_6(L, 1)$.

7. Consider a conformal transformation $L \to L_\epsilon = e^{-2\epsilon f} L$. Prove, that
   $$\frac{d}{d\epsilon}|_{\epsilon=0} a_{d-2}(e^{-2\epsilon f} F, L_\epsilon) = 0$$
   where F and f are smooth functions on M, $\dim M = d$.

8. (**Index Theorem**) Suppose that the Laplace type operators L_1 and L_2 can be factorized as products of first-order operators, $L_1 = D_1 D_2$ and $L_2 = D_2 D_1$.
(a) Prove that $a_k(L_1) = a_k(L_2)$ for $k \neq d$.
(b) What can you say about $a_d(L_1) - a_d(L_2)$?

9. Let D be the usual Dirac operator on a flat Euclidean space in an external electromagnetic field. Let $L = D^2$.
(a) Compute E, ω_μ and $\Omega_{\mu\nu}$.
(b) Compute the local chiral anomaly

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\left(\sigma(x)\gamma_5\, e^{-D^2/\Lambda^2}\right),$$

where $\sigma(x)$ is a smooth function of rapid decay.

10. Prove that the algebra of functions on noncommutative torus is associative.

11. Let $L = -(\partial_\mu^2 + L(\varphi\star\varphi) + R(\varphi\star\varphi) + L(\varphi)R(\varphi))$ act on scalar functions on the noncommutative torus $\mathbb{T}^2_\theta$ with a rational noncommutativity parameter. Compute $a_0(L)$, $a_2(L)$ and $a_4(L)$.